

Stability versions of Erdős-Ko-Rado type theorems, via isoperimetry

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Abstract

Erdős-Ko-Rado (EKR) type theorems yield upper bounds on the sizes of families of sets, subject to various intersection requirements on the sets in the family. ‘Stability’ versions of such theorems assert that if the size of a family is ‘close’ to the maximum possible size, then the family itself must be ‘close’ (in some appropriate sense) to a maximum-sized family.

In this paper, we present an approach to obtaining stability versions of EKR-type theorems, via isoperimetric inequalities for subsets of the hypercube. Our approach is rather general, and allows the leveraging of a wide variety of ‘exact’ EKR-type results into strong stability versions of these results, without going into the proofs of the original results.

We use this approach to obtain tight stability versions of the EKR theorem itself and of the Ahlswede-Khachatrian theorem on t -intersecting families of k -element subsets of $\{1, 2, \dots, n\}$ (for $k < \frac{n}{t+1}$), and to show that, somewhat surprisingly, all these results hold when the ‘intersection’ requirement is replaced by a much weaker requirement.

Other examples include stability versions of Frankl’s recent result on the Erdős matching conjecture, the Ellis-Films-Friedgut proof of the Simonovits-Sós conjecture, and various EKR-type results on r -wise (cross)- t -intersecting families.

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1 Introduction

1.1 Background

Let $[n] := \{1, 2, \dots, n\}$, and let $[n]^{(k)} := \{A \subset [n] : |A| = k\}$. A family $\mathcal{F} \subset \mathcal{P}([n])$ (i.e., a family of subsets of $[n]$) is said to be *intersecting* if for any $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$. For $t \in \mathbb{N}$, a family $\mathcal{F} \subset \mathcal{P}([n])$ is said to be *t-intersecting* if for any $A, B \in \mathcal{F}$, $|A \cap B| \geq t$.

A *dictatorship* is a family of the form $\{S \subset [n] : j \in S\}$ or $\{S \subset [n]^{(k)} : j \in S\}$ for some $j \in [n]$. A *t-umvirate* is a family of the form $\{S \subset [n] : B \subset S\}$ or $\{S \subset [n]^{(k)} : B \subset S\}$ for some $B \in [n]^{(t)}$.

One of the best-known theorems in extremal combinatorics is the Erdős-Ko-Rado (EKR) theorem [19]:

Theorem 1.1 (Erdős-Ko-Rado, 1961). *Let $k, n \in \mathbb{N}$ with $k < n/2$, and let $\mathcal{F} \subset [n]^{(k)}$ be an intersecting family. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Equality holds if and only if $\mathcal{F} = \{S \in [n]^{(k)} : j \in S\}$ for some $j \in [n]$.*

The Erdős-Ko-Rado theorem is the starting point of an entire subfield of extremal combinatorics, concerned with bounding the sizes of families of sets, under various intersection requirements on sets in the family. Such results are often called *EKR-type results*.

For more background and history on EKR-type results, we refer the reader to the surveys [12, 47] and the references therein. For our purposes here, we only mention the probably best-known EKR-type result — the Ahlswede-Khachatrian (AK) theorem [3], which bounds the size of *t-intersecting* families of *k*-element sets.

Theorem 1.2 (Ahlswede-Khachatrian, 1997). *For any $n, k, t, r \in \mathbb{N}$, let $\mathcal{F}_{n,k,t,r} := \{S \in [n]^{(k)} : |S \cap [t+2r]| \geq t+r\}$. Let $\mathcal{F} \subset [n]^{(k)}$ be a *t-intersecting* family. Then $|\mathcal{F}| \leq \max_r |\mathcal{F}_{n,k,t,r}|$.*

The $n \geq (t+1)(k-t+1)$ case of Theorem 1.2, in which $\max_r |\mathcal{F}_{n,k,t,r}| = |\mathcal{F}_{n,k,t,0}| = \binom{n-t}{k-t}$, was proved earlier, by Wilson [55]:

Theorem 1.3 (Wilson, 1984). *Let $n, k, t \in \mathbb{N}$ such that $n \geq (t+1)(k-t+1)$. Let $\mathcal{F} \subset [n]^{(k)}$ be a *t-intersecting* family. Then $|\mathcal{F}| \leq \binom{n-t}{k-t}$. If $n > (t+1)(k-t+1)$, then equality holds if and only if $\mathcal{F} = \{S \in [n]^{(k)} : B \subset S\}$ for some $B \in [n]^{(t)}$.*

Over the years, numerous authors have obtained *stability versions* of EKR-type results, asserting that if the size of a family is ‘close’ to the maximum possible size, then that family is ‘close’ (in some appropriate sense) to an extremal family.

Perhaps the first such ‘stability’ result was obtained in 1967 by Hilton and Milner [32]; they showed that if the size of an intersecting family is *very* close to $\binom{n-1}{k-1}$, then the family is *contained* in a dictatorship. A similar result for Wilson’s theorem was obtained in 1996 by Ahlswede and Khachatrian [2]. A simpler proof of the latter result was presented by Balogh and Mubayi [7], and an alternative result of the same class was obtained by Anstee and Keevash [5].

For families whose size is not very close to the maximum, Frankl [23] obtained in 1987 a rather strong stability version of the EKR theorem, proving that if an intersecting family \mathcal{F} satisfies $|\mathcal{F}| \geq (1-\epsilon)\binom{n-1}{k-1}$, then there exists a dictatorship \mathcal{D} such that $|\mathcal{F} \setminus \mathcal{D}| = O(\epsilon^{\log_{1-p} p} \binom{n}{k})$, where $p \approx k/n$. Frankl’s result, obtained using combinatorial ‘shifting’ (a.k.a. ‘compression’), is best-possible, and holds not only for $|\mathcal{F}|$ close to $\binom{n-1}{k-1}$ but rather for any $|\mathcal{F}| \geq 3\binom{n-2}{k-2} - 2\binom{n-3}{k-3}$. Proofs of somewhat weaker results using entirely different techniques were later presented by Dinur and Friedgut [14], Friedgut [28], and Keevash [38]. In [39], Keevash and Mubayi used Frankl’s result to prove an EKR-type theorem on set systems that do not contain a simplex or a

cluster. Recently, a different notion of stability for the EKR theorem was suggested by Bollobás, Narayanan and Raigorodskii in [10]; this has already been studied in several subsequent papers (e.g., [6, 11]).

In 2008, Friedgut [28] used spectral methods and Fourier analysis to obtain the following stability version of Wilson’s theorem:

Theorem 1.4 (Friedgut, 2008). *For any $t \in \mathbb{N}$ and any $\eta > 0$, there exists $c = c(t, \eta) > 0$ such that the following holds. Let $\eta n < k < (1/(t+1) - \eta)n$ and let $\epsilon \geq \sqrt{(\log n)/n}$. If $\mathcal{F} \subset [n]^{(k)}$ is a t -intersecting family with $|\mathcal{F}| \geq (1 - \epsilon)\binom{n-t}{k-t}$, then there exists $B \in [n]^{(t)}$ such that $|\mathcal{F} \setminus \mathcal{S}_B| \leq c\epsilon\binom{n}{k}$.*

Many other stability versions of EKR-type results have been obtained in recent years (see e.g. [25, 35, 40, 46, 52, 53]). Besides being interesting in their own right, such stability results often serve as a route for proving ‘exact’ EKR-type results. (In the more general setting of Turán-type problems, the idea of using a stability result to obtain an exact result goes back perhaps to Simonovits [50].)

1.2 Our results

In this paper, we present a new method for obtaining stability versions of EKR-type results. Our method, based on isoperimetric inequalities on the hypercube (see below), is rather general, and allows the leveraging of a wide range of ‘exact’ EKR-type results into stability results, without going into their proofs. It works whenever the following two conditions are satisfied:

- We have a ‘starting point’ — an exact EKR-type result for families $\mathcal{F} \subset [n]^{(k_0)}$ for some $k_0 \leq n$ (here, k_0 may depend on n).
- The extremal example is either a t -umvirate, or the ‘dual’ of a t -umvirate (equivalently, a family of the form $\{S \subset [n]^{(k_0)} : S \cap B \neq \emptyset\}$, for some $B \in [n]^{(t)}$).

Given these two conditions, our method allows us to deduce a strong stability result that holds for families $\mathcal{F} \subset [n]^{(k)}$ whenever $k \leq (1 - \eta)k_0$, for any fixed $\eta > 0$.

The two conditions above hold in a wide variety of settings, including:

1. The EKR theorem itself (where the extremal example is a dictatorship),
2. Wilson’s theorem on t -intersecting families (where the extremal example is a t -umvirate),
3. The Simonovits-Sós conjecture on triangle-intersecting families of graphs, proved recently by Ellis, Filmus and Friedgut [18] (where the extremal example is a specific type of 3-umvirate, namely all graphs containing a fixed triangle),
4. The Erdős matching conjecture on the maximal size of a family without s pairwise disjoint sets, proved recently by Frankl [24] for $n > (2s + 1)k - s$ (where the extremal example is the dual of an $(s - 1)$ -umvirate),
5. All known EKR-type results on r -wise (cross)- t -intersecting families (see e.g., [25, 35, 52, 53] and the references therein), where the extremal example is a t -umvirate.

In all these cases, our method leads to stability results which are much stronger than those obtained in several previous works (e.g., [14, 18, 28, 35, 53]).

For example, we obtain the following stability version of Wilson’s theorem:

Theorem 1.5. *For any $t \in \mathbb{N}$ and $\eta > 0$, there exists $\delta_0 = \delta_0(\eta, t) > 0$ such that the following holds. Let $k, n \in \mathbb{N}$ with $\frac{k}{n} \leq \frac{1}{t+1} - \eta$, and let $d \in \mathbb{N}$. Let $\mathcal{A} \subset [n]^{(k)}$ be a t -intersecting family with*

$$|\mathcal{A}| > \max \left\{ \binom{n-t}{k-t} (1 - \delta_0), \binom{n-t}{k-t} - \binom{n-t-d}{k-t} + (2^t - 1) \binom{n-t-d}{k-t-d+1} \right\}.$$

Then there exists a t -umvirate \mathcal{S}_B such that

$$|\mathcal{A} \setminus \mathcal{S}_B| \leq (2^t - 1) \binom{n-t-d}{k-t-d+1}.$$

Theorem 1.5 improves significantly over Theorem 1.4. Moreover, for d sufficiently large (as function of η, t), it is tight, up to replacing $2^t - 1$ with t , as evidenced by the families $(\mathcal{F}_{t,s})_{t,s \in \mathbb{N}}$, defined by

$$\begin{aligned} \mathcal{F}_{t,s} := & \left\{ A \subset [n]^{(k)} : [t] \subset A, \{t+1, \dots, t+s\} \cap A \neq \emptyset \right\} \\ & \cup \left\{ A \subset [n]^{(k)} : |[t] \cap A| = t-1, \{t+1, \dots, t+s\} \subset A \right\}. \end{aligned}$$

The p -biased setting

Following the works of Dinur and Friedgut [14] and Friedgut [28], we first obtain stability versions for EKR-type results in the so-called ‘biased-measure’ setting, and then leverage them to the ‘classical’ setting of k -element sets.

For $p \in [0, 1]$, the p -biased measure on $\mathcal{P}([n])$ is defined by

$$\mu_p(S) = p^{|S|} (1-p)^{n-|S|} \quad \forall S \subset [n].$$

In other words, we choose a random set by including each $j \in [n]$ independently with probability p . For $\mathcal{F} \subset \mathcal{P}([n])$, we define $\mu_p(\mathcal{F}) = \sum_{S \in \mathcal{F}} \mu_p(S)$. A family $\mathcal{F} \subset \mathcal{P}([n])$ is said to be *increasing* if it is closed under taking supersets, i.e. whenever $A \subset B$ and $A \in \mathcal{F}$, we have $B \in \mathcal{F}$.

The p -biased version of the EKR theorem is as follows.

Theorem 1.6 (Biased EKR Theorem). *Let $0 < p \leq 1/2$. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an intersecting family. Then $\mu_p(\mathcal{F}) \leq p$. If $p < 1/2$, then equality holds if and only if $\mathcal{F} = \{S \subset [n] : j \in S\}$ for some $j \in [n]$.*

First obtained by Ahlswede and Katona [1] in 1977, Theorem 1.6 was reproved numerous times using various different techniques (e.g., [13, 21, 22, 26]).

Over the years, p -biased versions of many EKR-type results have been obtained. In fact, it was shown that they can be deduced from the corresponding EKR-type results for k -element sets, with $k \approx pn$, using a method known as ‘going to infinity and back’ (see [15, 26]). For example, this method can be used to deduce the following biased version of Wilson’s theorem, from Theorem 1.3:

Theorem 1.7. *Let $\mathcal{F} \subset \mathcal{P}([n])$ be a t -intersecting family, and let $0 \leq p \leq 1/(t+1)$. Then $\mu_p(\mathcal{F}) \leq p^t$. If $p < 1/(t+1)$, equality holds if and only if \mathcal{F} is a t -umvirate.*

In the other direction, a biased EKR-type result implies only an approximate version of the corresponding ‘classical’ (k -uniform) result. However, in the cases we consider, we are able to leverage our biased-version stability results into more precise stability results in the classical setting. Furthermore, we show that these results hold when the ‘intersection’ requirement is replaced by a much weaker requirement.

Our proof-techniques

Informally, the basic idea behind our method is as follows. Instead of studying intersecting families, we consider the larger class of increasing families. (It is clear that in all biased EKR-type theorems, we can assume w.l.o.g. that the family is also increasing.) Given an increasing family $\mathcal{F} \subset \mathcal{P}([n])$, we view it as a subset of the hypercube $\{0,1\}^n$, and we compare the measures $\mu_p(\mathcal{F})$ for different values of p .

Russo's well-known lemma [49] asserts that the derivative of the function $f : p \mapsto \mu_p(\mathcal{F})$ satisfies $\frac{df}{dp}|_{p_0} = \mu_{p_0}(\partial\mathcal{F})$, where $\partial\mathcal{F}$ is the *edge boundary* of \mathcal{F} . (If $S \subset \{0,1\}^n$, the edge boundary ∂S of S is defined, as usual, to be the set of edges of the hypercube which join an element of S to an element of $\{0,1\}^n \setminus S$. We define $\mu_p(xy) = \mu_p(x) + \mu_p(y)$, for any hypercube edge xy .) Therefore, if for some $p_1 < p_0$ we know that $\mu_{p_0}(\mathcal{F})$ is 'not much larger' than $\mu_{p_1}(\mathcal{F})$, it follows from the Mean Value theorem that there exists $p_2 \in (p_1, p_0)$ such that the edge boundary of \mathcal{F} is 'small' with respect to μ_{p_2} , i.e., $\mu_{p_2}(\partial\mathcal{F})$ is 'small'.

On the other hand, the biased version of the edge-isoperimetric inequality on the hypercube (Theorem 2.2 below; see e.g. [34]) asserts that for any $p \in (0,1)$ and any increasing $\mathcal{F} \subset \mathcal{P}([n])$, $\mu_p(\partial\mathcal{F})$ cannot be 'too small' as function of $\mu_p(\mathcal{F})$, and the minimum is attained (for any p) only by t -umvirates. As for a t -umvirate \mathcal{S}_B we have $\mu_p(\mathcal{S}_B) = p^t$ for any $p \in (0,1)$, we can deduce that if an increasing family $\mathcal{F} \subset \mathcal{P}([n])$ satisfies $\mu_{p_0}(\mathcal{F}) \leq p_0^t$ for some p_0, t , then for any $p_1 < p_0$, we have $\mu_{p_1}(\mathcal{F}) \leq p_1^t$.

Furthermore, we use a stability version of the biased edge-isoperimetric inequality [17] to deduce that if $\mu_{p_0}(\mathcal{F}) \leq p_0^t$, and also $\mu_{p_1}(\mathcal{F}) \geq p_1^t(1-\epsilon)$, then \mathcal{F} must be 'close' to a t -umvirate.

The argument sketched above yields a 'rough' stability result. We leverage it into a more precise stability result using a bootstrapping technique which is also based upon an isoperimetric argument. As our main biased stability theorem (Theorem 3.1) is somewhat technical, we delay its statement until Section 3. Here we present a special case (the 'dictatorship' case), whose statement is simpler:

Theorem 1.8. *There exist absolute constants $C, c > 0$ such that the following holds. Let $0 < p < 1/2$, and let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family such that $\mu_{1/2}(\mathcal{F}) \leq 1/2$ and*

$$\mu_p(\mathcal{F}) \geq \min\{Cp^2, p(1 - c(\frac{1}{2} - p))\}. \quad (1)$$

Let $\epsilon > 0$. If

$$\mu_p(\mathcal{F}) \geq p \left(1 - \epsilon^{\log_p(1-p)}\right) + (1-p)\epsilon, \quad (2)$$

then $\mu_p(\mathcal{F} \setminus \mathcal{F}_j) \leq (1-p)\epsilon$ for some dictatorship \mathcal{F}_j .

Theorem 1.8 is tight for the families $\{\tilde{\mathcal{G}}_i\}_{i \geq 3}$, defined by:

$$\tilde{\mathcal{G}}_i = \{A \subset [n] : (1 \in A) \wedge (A \cap \{2, 3, \dots, i\} \neq \emptyset)\} \cup \{A \subset [n] : (1 \notin A) \wedge (\{2, 3, \dots, i\} \subset A)\}.$$

Indeed, we have $\mu_p(\tilde{\mathcal{G}}_i) = p(1 - (1-p)^{i-1}) + (1-p)p^{i-1}$, which corresponds to Condition (2) with $\epsilon = p^{i-1}$, and $\mu_p(\tilde{\mathcal{G}}_i \setminus \mathcal{F}_1) = (1-p)p^{i-1} = (1-p)\epsilon$.

Note that any intersecting family \mathcal{F} satisfies $\mu_{1/2}(\mathcal{F}) \leq 1/2$ since it cannot contain both a set and its complement. Hence, Theorem 1.8 implies a stability version of the biased EKR theorem (i.e., Theorem 1.6 above). Moreover, this stability version is tight, as the families $\tilde{\mathcal{G}}_i$ are intersecting.

Hence, not only do we obtain a tight stability version of the biased EKR theorem; we also show that the result holds under a significantly weaker condition. Indeed, instead of requiring

that the family is intersecting, it is sufficient to require that it is contained in an increasing family \mathcal{F} which satisfies $\mu_{1/2}(\mathcal{F}) \leq 1/2$. This condition holds for many families that are far from satisfying any intersection property.

Furthermore, our stability version is ‘strong’, in the sense that for small p , it holds not only when $\mu_p(\mathcal{F})$ is close to p (which is the maximal possible value), but rather for any $\mu_p(\mathcal{F}) > Cp^2$, as apparent from Condition (1).

In a similar way, we obtain a strong stability version of the p -biased Wilson theorem, one which holds not only for t -intersecting families, but also for any increasing family $\mathcal{F} \subset \mathcal{P}([n])$ such that $\mu_{1/(t+1)}(\mathcal{F}) \leq (t+1)^{-t}$.

We also obtain a ‘dual’ version (Theorem 3.19) of our main biased stability theorem, and we use it to obtain a stability result for the biased version of the Erdős matching conjecture.

With the biased-measure results in hand, we turn to prove our stability results in the classical setting of k -element sets, the most important of these being Theorem 1.5 and a stability result for the Erdős matching conjecture (Theorem 4.12). These results are of course based on a reduction to the biased-measure results, but the reduction is not straightforward, and requires several technical results on cross-intersecting families, which we prove in Section 4.

Organization of the paper

In Section 2, we give some definitions and notation, and present some of the main tools and prior results used in our paper. In Section 3, we prove our main theorems in the biased-measure setting and present several applications. In Section 4, we prove our stability results in the classical setting of k -element sets. In Section 5, we compare our results with some prior results on intersecting and t -intersecting families. We conclude the paper with some open problems, in Section 6.

2 Definitions and tools

2.1 Definitions and notation

If $\mathcal{F} \subset \mathcal{P}([n])$, we define the *dual* family \mathcal{F}^* by $\mathcal{F}^* = \{[n] \setminus A : A \notin \mathcal{F}\}$. We write $\mathcal{F}^c := \mathcal{P}([n]) \setminus \mathcal{F}$ and $\overline{\mathcal{F}} := \{[n] \setminus A : A \in \mathcal{F}\}$, so that $\mathcal{F}^* = (\overline{\mathcal{F}})^c = \overline{\mathcal{F}^c}$. We denote by \mathcal{F}^\uparrow the up-closure of \mathcal{F} , i.e. the minimal increasing subfamily of $\mathcal{P}([n])$ which contains \mathcal{F} . If $\mathcal{F} \subset \mathcal{P}([n])$ and $C \subset B \subset [n]$, we define $\mathcal{F}_B^C := \{S \in \mathcal{P}([n] \setminus B) : S \cup C \in \mathcal{F}\}$. If $B \subset [n]$, we write $\mathcal{S}_B := \{A \subset [n] : B \subset A\}$, and we write $\text{OR}_B := (\mathcal{S}_B)^* = \{A \subset [n] : A \cap B \neq \emptyset\}$.

Let Q_n denote the graph of the n -dimensional hypercube, i.e. the graph with vertex-set $\{0, 1\}^n$, where two vertices are joined by an edge if they differ in exactly one coordinate. If $A \subset \{0, 1\}^n$, we let ∂A denote the *edge-boundary* of the set A w.r.t. Q_n , meaning the set of edges of Q_n which join a vertex in A to a vertex in $\{0, 1\}^n \setminus A$. We will often identify $\{0, 1\}^n$ with $\mathcal{P}([n])$, via the correspondence $(x_1, x_2, \dots, x_n) \leftrightarrow \{i \in [n] : x_i = 1\}$. Hence, we say $A \subset \{0, 1\}^n$ is increasing if the corresponding subset of $\mathcal{P}([n])$ is increasing, i.e. if whenever $(x_1, \dots, x_n) \in A$ and $y_i \geq x_i$ for all $i \in [n]$, we have $(y_1, \dots, y_n) \in A$.

We write 1_A for the *indicator function* of A , i.e., the Boolean function

$$1_A : \{0, 1\}^n \rightarrow \{0, 1\}; \quad 1_A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A. \end{cases}$$

A *subcube* of $\{0,1\}^n$ is a set of the form $\{x \in \{0,1\}^n : x_i = c_i \ \forall i \in T\}$, where $T \subset [n]$ and $c_i \in \{0,1\}$ for each $i \in T$. Under the identification above, a t -umvirate corresponds to an $(n-t)$ -dimensional, increasing subcube.

For each $i \in [n]$, we let $e_i = (0,0,\dots,0,1,0,\dots,0)$ denote the i th unit vector. For $x, y \in \{0,1\}^n$, we let $x + y$ denote the sum of x and y modulo 2.

Let $0 \leq p \leq 1$. By identifying $\{0,1\}^n$ with $\mathcal{P}([n])$ as above, the p -biased measure on $\mathcal{P}([n])$ (defined in the Introduction) can alternatively be defined on $\{0,1\}^n$:

$$\mu_p(A) = \sum_{x \in A} p^{|\{i \in [n] : x_i = 1\}|} (1-p)^{|\{i \in [n] : x_i = 0\}|} \quad \forall A \subset \{0,1\}^n.$$

As usual, if $f : \{0,1\}^n \rightarrow \mathbb{R}$, we let $\mu_p[f]$ denote the expectation of f with respect to the measure μ_p .

We say that a function $f : \{0,1\}^n \rightarrow \mathbb{R}$ is *increasing* if whenever $x, y \in \{0,1\}^n$ with $x_i \leq y_i$ for all $i \in [n]$, we have $f(x) \leq f(y)$.

If $f : \{0,1\}^n \rightarrow \{0,1\}$ is a Boolean function, we define the *influence of f in direction i* (with respect to μ_p) by

$$\text{Inf}_i[f] := \mu_p(\{x \in \{0,1\}^n : f(x) \neq f(x + e_i)\}).$$

We define the *total influence* of f (w.r.t. μ_p) by $I_p[f] := \sum_{i=1}^n \text{Inf}_i[f]$.

Similarly, if $A \subset \{0,1\}^n$, we define the *influence of A in direction i* (w.r.t. μ_p) by $\text{Inf}_i[A] := \text{Inf}_i[1_A]$, and we define *total influence* of A (w.r.t. μ_p) by $I_p[A] := I_p[1_A]$.

Throughout the paper, we use the letters C, c to denote positive constants (possibly depending upon some parameters); the values of these constants and the parameters on which they are allowed to depend may differ between different theorems, propositions etc.

2.2 Background and tools

The edge-isoperimetric inequality on the hypercube

Harper [30], Lindsay [45], Bernstein [8] and Hart [31] solved the edge-isoperimetric problem for Q_n , showing that among all subsets of $\{0,1\}^n$ of fixed size, initial segments of the binary ordering on $\{0,1\}^n$ have the smallest edge-boundary. (The *binary ordering* on $\{0,1\}^n$ is defined by $x < y$ iff $\sum_{i=1}^n 2^{i-1}x_i < \sum_{i=1}^n 2^{i-1}y_i$.) The following weaker (but still useful) statement has an easy proof by induction on n .

Theorem 2.1. *If $A \subset \{0,1\}^n$, then*

$$|\partial A| \geq |A| \log_2(2^n/|A|). \quad (3)$$

Equality holds in (3) if and only if A is a subcube.

The following analogue of Theorem 2.1 for the p -biased measure is well-known; it appears for example in [34].

Theorem 2.2. *Suppose that $0 < p < 1$ and $A \subset \{0,1\}^n$ is increasing, or alternatively that $0 < p \leq 1/2$ and $A \subset \{0,1\}^n$ is arbitrary. Then*

$$pI_p[A] \geq \mu_p(A) \log_p(\mu_p(A)). \quad (4)$$

If $0 < p < 1$ and A is increasing, then equality holds in (4) if and only if A is an increasing subcube (i.e., a t -umvirate for some $t \in \mathbb{N}$).

Like Theorem 2.1, this can be proved by a straightforward induction on n . We note that (4) does not hold for arbitrary subsets $A \subset \{0,1\}^n$ when $p > 1/2$; indeed, the ‘antidictatorship’ $A = \{x \in \{0,1\}^n : x_1 = 0\}$ is a counterexample.

A stability version of the biased edge-isoperimetric inequality on the hypercube

In [16], the first author proved the following.

Theorem 2.3. *There exists an absolute constant $c_0 > 0$ such that the following holds. Let $0 \leq \epsilon \leq c_0$. If $A \subset \{0, 1\}^n$ with $|\partial A| \leq |A|(\log_2(2^n/|A|) + \epsilon)$, then there exists a subcube $C \subset \{0, 1\}^n$ such that*

$$|A \Delta C| \leq \frac{2\epsilon}{\log_2(1/\epsilon)} |A|.$$

Theorem 2.3 says that a subset of $\{0, 1\}^n$ whose edge-boundary has size is ‘close’ to the bound (3) must be ‘close’ in symmetric difference to a subcube. The proof uses a lower bound of Talagrand [51] concerning the vector of influences, obtained by Fourier-analytic techniques.

In this paper, we need the following ‘ p -biased’ analogue of Theorem 2.3 for increasing Boolean functions, which we prove in [17].

Theorem 2.4. *Let $\eta > 0$. There exist $C_1 = C_1(\eta)$, $c_0 = c_0(\eta) > 0$ such that the following holds. Let $0 < p \leq 1 - \eta$, and let $0 \leq \epsilon \leq c_0/\ln(1/p)$. Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be an increasing Boolean function such that*

$$pI_p[f] \leq \mu_p[f] (\log_p(\mu_p[f]) + \epsilon).$$

Then f is $\frac{C_1 \epsilon \ln(1/p)}{\ln\left(\frac{1}{\epsilon \ln(1/p)}\right)} \mu_p[f]$ -close (w.r.t. μ_p) to the indicator function of an increasing subcube.

Our proof of Theorem 2.4 in [17] follows a similar approach to the proof of Theorem 2.3 in [16], but is somewhat simpler and uses only elementary techniques (it does not rely on any result proved using Fourier analysis, for example).

Russo’s Lemma and a combination with the edge-isoperimetric inequality

Russo’s Lemma [49] relates the derivative of the function $p \mapsto \mu_p(A)$ to the total influence $I_p(A)$, where $A \subset \{0, 1\}^n$ is increasing.

Lemma 2.5 (Russo’s Lemma). *Let $A \subset \{0, 1\}^n$ be increasing, and let $0 < p_0 < 1$. Then*

$$\left. \frac{d\mu_p(A)}{dp} \right|_{p=p_0} = I_{p_0}[A].$$

We will make crucial use of the following consequence of Russo’s Lemma and Theorem 2.2 (this appears e.g. in [29], Theorem 2.38).

Lemma 2.6. *Let $A \subset \{0, 1\}^n$ be increasing. Then the function $p \mapsto \log_p(\mu_p(A))$ is monotone non-increasing on $(0, 1)$. If A is not a subcube, then this function is strictly monotone decreasing on $(0, 1)$.*

Proof. Let $f(p) := \log_p(\mu_p(A)) = \ln \mu_p(A) / \ln p$. We have

$$\frac{df}{dp} = \frac{\frac{1}{\mu_p(A)} \frac{d\mu_p(A)}{dp} \ln p - \frac{1}{p} \ln(\mu_p(A))}{(\ln p)^2} = \frac{\frac{1}{\mu_p(A)} I_p[A] \ln p - \frac{1}{p} \ln(\mu_p(A))}{(\ln p)^2} \leq 0, \quad (5)$$

using Russo’s Lemma and (4). If A is not a subcube, then strict inequality holds in (4) for all $p \in (0, 1)$, so strict inequality holds in (5), and therefore f is strictly monotone decreasing. \square

The following useful ‘monotonicity’ lemma is an easy consequence of Lemma 2.6.

Lemma 2.7. *Let $p_0 \in (0, 1)$, $p \in (0, p_0)$ and $t > 0$. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family.*

1. *Suppose $\mu_{p_0}(\mathcal{F}) \leq p_0^t$. Then $\mu_p(\mathcal{F}) \leq p^t$, with equality if and only if $t \in \mathbb{N}$ and \mathcal{F} is a t -umvirate.*
2. *Suppose $\mu_{p_0}(\mathcal{F}) \leq 1 - (1 - p_0)^t$. Then $\mu_p(\mathcal{F}) \leq 1 - (1 - p)^t$, with equality if and only if $t \in \mathbb{N}$ and $\mathcal{F} = \text{OR}_B$ for some $B \in [n]^{(t)}$.*

Proof. The upper bound in item (1) follows immediately from Lemma 2.6. For the equality part, suppose that $\mathcal{F} \subset \mathcal{P}([n])$ is not an increasing subcube. Then the function f defined in the proof of Lemma 2.6 (of course, setting $A := \mathcal{F}$) is strictly decreasing, and therefore $\mu_p(\mathcal{F}) < p^t$ for any $p \in (0, p_0)$. Hence, if $\mu_p(\mathcal{F}) = p^t$, then \mathcal{F} must be an increasing subcube; it must therefore be a t -umvirate.

To prove item (2), let \mathcal{F}^* be the dual of \mathcal{F} . Then

$$\mu_{1-p_0}(\mathcal{F}^*) = \mu_{p_0}(\mathcal{F}^c) \geq (1 - p_0)^t.$$

Since \mathcal{F}^* is increasing, we may apply item (1) for $1 - p \geq 1 - p_0$, obtaining

$$1 - \mu_p(\mathcal{F}) = \mu_p(\mathcal{F}^c) = \mu_{1-p}(\mathcal{F}^*) \geq (1 - p)^t.$$

The assertion follows by rearranging. The equality part follows exactly as in item (1). \square

Note that the $t = 1$, $p_0 = 1/2$ case of Lemma 2.7 (1), immediately yields the following strengthening of the biased EKR theorem:

Corollary 2.8. *Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family such that $\mu_{1/2}(\mathcal{F}) \leq 1/2$. Then for any $p \in (0, 1/2)$, $\mu_p(\mathcal{F}) \leq p$. Equality holds if and only if \mathcal{F} is a dictatorship.*

It is perhaps somewhat surprising that the conclusion of the biased EKR theorem holds under a considerably weaker hypothesis than being intersecting.

We will also need the following consequence of Lemma 2.6.

Lemma 2.9. *Let $0 < p \leq 1/2$. Suppose $\mathcal{F}, \mathcal{G} \subset \mathcal{P}([n])$ are increasing and cross-intersecting (meaning that $A \cap B \neq \emptyset$ whenever $A \in \mathcal{F}$ and $B \in \mathcal{G}$). Then*

$$\mu_p(\mathcal{G}) \leq (1 - \mu_p(\mathcal{F}))^{\log_{1-p}(p)}.$$

Proof. Since \mathcal{F} and \mathcal{G} are cross-intersecting, we have $\mathcal{G} \subset \mathcal{F}^*$. Hence, $\mu_{1-p}(\mathcal{G}) \leq \mu_{1-p}(\mathcal{F}^*) = 1 - \mu_{1-p}(\mathcal{F}) = 1 - \mu_p(\mathcal{F})$. Hence, by Lemma 2.6, $\mu_p(\mathcal{G}) \leq (\mu_{1-p}(\mathcal{G}))^{\log_{1-p}(p)} \leq (1 - \mu_p(\mathcal{F}))^{\log_{1-p}(p)}$. \square

Shadows of t -intersecting families

The *lower shadow* of a family $\mathcal{F} \subset [n]^{(k)}$ is defined as $\partial(\mathcal{F}) = \{A \in [n]^{(k-1)} : \exists B \in \mathcal{F}, A \subset B\}$. Similarly, the *s-shadow* of \mathcal{F} is defined as $\partial^s(\mathcal{F}) = \{A \in [n]^{(k-s)} : \exists B \in \mathcal{F}, A \subset B\}$. The shadow is a central notion in extremal combinatorics that appears in a multitude of techniques and results (e.g., the classical Kruskal-Katona theorem [37, 44]).

As we replace families of k -element subsets with increasing families, we define a notion of shadow for them. For an increasing family $\mathcal{F} \subset \mathcal{P}([n])$, we define the s -shadow of \mathcal{F} by

$$\partial^s(\mathcal{F}) := \{A \in \mathcal{P}([n]) : \exists C \in [n]^{(s)}, A \cup C \in \mathcal{F}\}.$$

Note that in this definition we do *not* require $A \cap C = \emptyset$, so our definition of the s -shadow of an increasing family is weaker than the above definition of the s -shadow of a k -uniform family; however, it is the ‘right’ notion for our purposes. We use the same notation for each notion; the one we mean will always be clear from the context.

We use the following classical theorem of Katona [36] on t -shadows of t -intersecting subsets of $[n]^{(k)}$.

Theorem 2.10 (Katona’s ‘shadow/intersection theorem’, 1964). *Let $\mathcal{F} \subset [n]^{(k)}$ be a t -intersecting family. Then $|\partial^t(\mathcal{F})| \geq |\mathcal{F}|$.*

We also use the following analogue of Theorem 2.10 for increasing subsets of $\mathcal{P}([n])$.

Theorem 2.11. *Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing t -intersecting family. Then for any $0 < p < 1$,*

$$\mu_p(\partial^t(\mathcal{F})) \geq \frac{(1-p)^t}{p^t} \mu_p(\mathcal{F}).$$

Proof. We deduce this from Theorem 2.10 using the Dinur-Safra / Frankl-Tokushige method of ‘going to infinity and back’ (see [15, 26]). For a family $\mathcal{F} \subset \mathcal{P}([n])$ and $N > n$, we define

$$\mathcal{F}_{N,k} := \{A \in [N]^{(k)} : A \cap [n] \in \mathcal{F}\}.$$

We claim that

$$\partial^t(\mathcal{F}_{N,k}) \subset (\partial^t(\mathcal{F}))_{N,k-t}. \quad (6)$$

Indeed, if $A \in \partial^t(\mathcal{F}_{N,k})$, then there exists $C \in [N]^{(t)}$ such that $A \cap C = \emptyset$ and $A \cup C \in \mathcal{F}_{N,k}$. Hence, $(A \cap [n]) \cup (C \cap [n]) = (A \cup C) \cap [n] \in \mathcal{F}$. Since \mathcal{F} is increasing, for any $C' \in [n]^{(t)}$ with $C \cap [n] \subset C'$, we have $(A \cap [n]) \cup C' \in \mathcal{F}$. Therefore, $A \cap [n] \in \partial^t(\mathcal{F})$, and thus, $A \in (\partial^t(\mathcal{F}))_{N,k-t}$.

As $\mathcal{F} \subset \mathcal{P}([n])$ is increasing and t -intersecting, for any $N > k > n$, $\mathcal{F}_{N,k}$ is also t -intersecting. Hence, by Theorem 2.10 we have $|\partial^t(\mathcal{F}_{N,k})| \geq |\mathcal{F}_{N,k}|$. Combining this with (6), we get

$$|\mathcal{F}_{N,k}| \leq |\partial^t(\mathcal{F}_{N,k})| \leq |(\partial^t(\mathcal{F}))_{N,k-t}|.$$

Now, it is easily checked that for any $\mathcal{F} \subset \mathcal{P}([n])$,

$$\mu_p(\mathcal{F}) = \lim_{N \rightarrow \infty} \frac{|\mathcal{F}_{N,\lfloor pN \rfloor}|}{\binom{N}{\lfloor pN \rfloor}} = \lim_{N \rightarrow \infty} \frac{|\mathcal{F}_{N,\lfloor pN \rfloor - t}|}{\binom{N}{\lfloor pN \rfloor - t}}.$$

Hence,

$$\begin{aligned} \mu_p(\partial^t(\mathcal{F})) &= \lim_{N \rightarrow \infty} \frac{|(\partial^t(\mathcal{F}))_{N,\lfloor pN \rfloor - t}|}{\binom{N}{\lfloor pN \rfloor - t}} \geq \lim_{N \rightarrow \infty} \frac{|\mathcal{F}_{N,\lfloor pN \rfloor}|}{\binom{N}{\lfloor pN \rfloor - t}} = \left(\lim_{N \rightarrow \infty} \frac{\binom{N}{\lfloor pN \rfloor}}{\binom{N}{\lfloor pN \rfloor - t}} \right) \left(\lim_{N \rightarrow \infty} \frac{|\mathcal{F}_{N,\lfloor pN \rfloor}|}{\binom{N}{\lfloor pN \rfloor}} \right) \\ &= \left(\frac{1-p}{p} \right)^t \lim_{N \rightarrow \infty} \frac{|\mathcal{F}_{N,\lfloor pN \rfloor}|}{\binom{N}{\lfloor pN \rfloor}} = \left(\frac{1-p}{p} \right)^t \mu_p(\mathcal{F}), \end{aligned}$$

as asserted. \square

3 The main ‘biased’ stability theorem

In this section, we state and prove our main biased-measure stability theorem, and present several of its applications.

Theorem 3.1. *Let $t \in \mathbb{N}$ and let $0 < p_0 < 1$. Then there exist $C = C(p_0, t)$, $c = c(p_0, t) > 0$ such that the following holds. Let $0 < p < p_0$, and let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family such that $\mu_{p_0}(\mathcal{F}) \leq p_0^t$ and*

$$\mu_p(\mathcal{F}) \geq \min\{Cp^{t+1}, p^t(1 - c(p_0 - p))\}. \quad (7)$$

Let $\epsilon > 0$, and define $\tilde{c} := \left(\frac{1-p_0}{p_0}\right)^{\log_{1-p_0}(1-p)}$. If

$$\mu_p(\mathcal{F}) \geq p^t \left(1 - \tilde{c}\epsilon^{\log_p(p_0) \log_{1-p_0}(1-p)}\right) + (1-p)p^{t-1}\epsilon, \quad (8)$$

then there exists a t -umvirate \mathcal{S}_B such that

$$\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq (1-p)p^{t-1}\epsilon. \quad (9)$$

Remark 3.2. It will follow from our proof that for any $\xi \in (0, 1/2]$,

$$\sup_{p_0 \in [\xi, 1-\xi]} C(p_0, t) = O_{\xi, t}(1), \quad \inf_{p_0 \in [\xi, 1-\xi]} c(p_0, t) = \Omega_{\xi, t}(1).$$

Theorem 3.1 is tight for infinitely many of the families $\{\tilde{\mathcal{H}}_{t,s,r} : t, s, r \in \mathbb{N}\}$, defined by:

$$\begin{aligned} \tilde{\mathcal{H}}_{t,s,r} = & \{A \subset [n] : ([t] \subset A) \wedge (A \cap \{t+1, t+2, \dots, t+r\} \neq \emptyset)\} \\ & \cup \{A \subset [n] : ([t-1] \subset A) \wedge (t \notin A) \wedge (\{t+1, t+2, \dots, t+s\} \subset A)\}. \end{aligned}$$

To see this, for each $(t, s, r) \in \mathbb{N}^3$ with $r, s \geq 2$, choose the unique $p_0 \in (0, 1)$ such that $(1-p_0)^{r-1} = p_0^{s-1}$. Then we have $\mu_{p_0}(\mathcal{H}_{t,s,r}) = p_0^t$, $\mu_p(\mathcal{H}_{t,s,r}) = p^t(1 - (1-p)^r) + (1-p)p^{t+s-1}$, and $\mu_p(\mathcal{H}_{t,s,r} \setminus \mathcal{S}_{[t]}) = (1-p)p^{t+s-1}$, so equality holds in (8) and (9) with $\epsilon = p^s$. Provided r is sufficiently large, the condition (7) is also satisfied.

Roughly speaking, Theorem 3.1 asserts that if $\mu_{p_0}(\mathcal{F}) \leq p_0^t$ (this replaces the t -intersection condition) and $\mu_p(\mathcal{F})$ is ‘large enough’, then the following stability result holds: if $\mu_p(\mathcal{F}) \geq p^t(1 - \epsilon')$, then there exists a t -umvirate \mathcal{S}_B such that $\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \lesssim O(\epsilon'^{\log_{p_0}(p) \log_{1-p}(1-p_0)})$. The meaning of ‘large enough’, reflected by Condition (7), differs between the cases of ‘large’ p and ‘small’ p . When $p = \Omega(p_0)$, the stability theorem requires that $\mu_p(\mathcal{F})$ is close to p^t . For $p \ll p_0$, the assertion is stronger, saying that the conclusion holds once $\mu_p(\mathcal{F})$ is significantly larger than p^{t+1} (even if it is far from p^t).

Since the proof of the theorem is somewhat complex, we first present the three components of the proof separately, and then we show how combining them yields the theorem. As in [14, 28], we distinguish between the cases of ‘large p ’, i.e., $p = \Omega_{p_0}(1)$, and of ‘small p ’, i.e., $p \ll p_0$, and handle each case differently.

The following lemma will be used repeatedly.

Lemma 3.3. *Let $p_0, \delta \in (0, 1)$, let $t, n \in \mathbb{N}$, and let $0 < p < p_0$. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family with*

$$\mu_p(\mathcal{F} \setminus \mathcal{S}_{[t]}) \geq (1-p)p^{t-1}\delta.$$

Then:

(a)

$$\mu_{p_0}(\mathcal{F} \setminus \mathcal{S}_{[t]}) \geq (1 - p_0) p_0^{t-1} \left(\delta^{\log_p p_0} \right).$$

(b) If, in addition, $\mu_{p_0}(\mathcal{F}) \leq p_0^t$, then

$$\mu_p(\mathcal{F} \cap \mathcal{S}_{[t]}) \leq p^t \left(1 - \tilde{c} \delta^{\log_p p_0 \log_{1-p}(1-p_0)} \right),$$

$$\text{where } \tilde{c} := \left(\frac{1-p_0}{p_0} \right)^{\log_{1-p_0}(1-p)} = \frac{1-p}{p_0^{\log_{1-p_0}(1-p)}}.$$

Proof. To prove (a), we note that the following equations hold.

$$\begin{aligned} \mu_p(\mathcal{F} \setminus \mathcal{S}_{[t]}) &= \sum_{i \in [t]} (1-p) p^{i-1} \mu_p(\mathcal{F}_{[i]}^{[i-1]}), \\ \mu_{p_0}(\mathcal{F} \setminus \mathcal{S}_{[t]}) &= \sum_{i \in [t]} (1-p_0) p_0^{i-1} \mu_{p_0}(\mathcal{F}_{[i]}^{[i-1]}). \end{aligned}$$

By Lemma 2.7(1),

$$\mu_{p_0}(\mathcal{F}_{[i]}^{[i-1]}) \geq \mu_p(\mathcal{F}_{[i]}^{[i-1]})^{\log_p p_0}.$$

Therefore,

$$\begin{aligned} \mu_{p_0}(\mathcal{F} \setminus \mathcal{S}_{[t]}) &\geq \sum_{i \in [t]} (1-p_0) p_0^{i-1} \mu_p(\mathcal{F}_{[i]}^{[i-1]})^{\log_p p_0} \\ &= \frac{(1-p_0)}{(1-p)^{\log_p p_0}} \sum_{i \in [t]} \left((1-p) p^{i-1} \mu_p(\mathcal{F}_{[i]}^{[i-1]}) \right)^{\log_p p_0} \\ &\geq \frac{(1-p_0)}{(1-p)^{\log_p p_0}} (\mu_p(\mathcal{F} \setminus \mathcal{S}_{[t]}))^{\log_p p_0} \geq (1-p_0) p_0^{t-1} \delta, \end{aligned}$$

where the penultimate inequality holds since for any non-negative a_1, \dots, a_ℓ and for any $0 \leq \beta \leq 1$, we have $(\sum a_i)^\beta \leq \sum a_i^\beta$, and the last inequality uses the assumption on $\mu_p(\mathcal{F} \setminus \mathcal{S}_{[t]})$. This completes the proof of (a).

To prove (b), we note that (a) and the assumption $\mu_{p_0}(\mathcal{F}) \leq p_0^t$ imply

$$\mu_{p_0}(\mathcal{F}_{[t]}^{[t]}) = \frac{\mu_{p_0}(\mathcal{F} \cap \mathcal{S}_{[t]})}{p_0^t} \leq 1 - \delta^{\log_p p_0} \frac{(1-p_0)}{p_0}.$$

By Lemma 2.7(2), this implies:

$$\mu_p(\mathcal{F} \cap \mathcal{S}_{[t]}) = p^t \mu_p(\mathcal{F}_{[t]}^{[t]}) \leq p^t \left(1 - \tilde{c} \delta^{\log_p p_0 \log_{1-p}(1-p_0)} \right),$$

as asserted. □

3.1 A rough stability result for large p

In this subsection we consider the case $p \geq \zeta p_0$, for a constant $\zeta > 0$ that will be chosen later. (Meanwhile, we state the results in terms of ζ .)

The following proposition shows that if $\mu_p(\mathcal{F})$ is sufficiently close to p^t , then \mathcal{F} is ‘somewhat’ close to a t -umvirate. We will use it in the proof of Theorem 3.1 as the basis of a bootstrapping process.

Proposition 3.4. *For any $\eta \in (0, 1)$, there exists $C = C(\eta) > 0$ such that the following holds. Let $\zeta \in (0, 1)$, let $\zeta p_0 \leq p \leq p_0 \leq 1 - \eta$, let $t \in \mathbb{N}$, and let $0 < \epsilon \leq 1$. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family such that $\mu_{p_0}(\mathcal{F}) \leq p_0^t$ and $\mu_p(\mathcal{F}) \geq p^t(1 - \epsilon')$, where $\epsilon' := \frac{C\epsilon(p_0 - p)}{\ln\left(\frac{1}{\zeta p_0}\right)^t}$.*

Then there exists $B \in [n]^{(t)}$ such that $\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq \epsilon p^{t-1}(1 - p)$.

Proof. The proof of the proposition proceeds in three steps. First, we use Russo’s lemma (i.e., Lemma 2.5) to show that for some $p' \in (p_0, p)$, the total influence $I_{p'}(\mathcal{F})$ is very close to the total influence of a t -umvirate. Then, we use the stability version of the isoperimetric inequality (i.e., Theorem 2.4) to deduce that there exists $B \in [n]^{(t)}$ such that $\mu_{p'}(\mathcal{F} \setminus \mathcal{S}_B)$ is small. Finally, we use Lemma 3.3 to complete the proof.

By Russo’s lemma, we have

$$\begin{aligned} \int_p^{p_0} I_{p'}(\mathcal{F}) dp' &= \mu_{p_0}(\mathcal{F}) - \mu_p(\mathcal{F}) \leq p_0^t - p^t + \epsilon' p^t, \\ \int_p^{p_0} I_{p'}(\mathcal{S}_{[t]}) dp' &= \mu_{p_0}(\mathcal{S}_{[t]}) - \mu_p(\mathcal{S}_{[t]}) = p_0^t - p^t. \end{aligned}$$

Subtracting the equations, we get

$$\int_p^{p_0} (I_{p'}(\mathcal{F}) - I_{p'}(\mathcal{S}_{[t]})) dp' \leq \epsilon' p^t.$$

Hence, there exists $p' \in (p, p_0)$ such that

$$I_{p'}(\mathcal{F}) - I_{p'}(\mathcal{S}_{[t]}) \leq \frac{\epsilon' p^t}{p_0 - p}. \quad (10)$$

In addition, by the assumption on $\mu_p(\mathcal{F})$ and $\mu_{p_0}(\mathcal{F})$ and by Lemma 2.7, we have:

$$p^{tt} \geq \mu_{p'}(\mathcal{F}) \geq (p^t(1 - \epsilon'))^{\log_p p'} \geq p^{tt}(1 - \epsilon'). \quad (11)$$

Using (10), (11), and the fact that $I_{p'}(\mathcal{S}_{[t]}) = t(p')^{t-1}$, we obtain:

$$\begin{aligned}
p' I_{p'}(\mathcal{F}) &\leq p' \left(t p'^{t-1} + \frac{\epsilon' p^t}{p_0 - p} \right) \leq p'^t \left(\log_{p'}(\mu_{p'}(\mathcal{F})) + \frac{\epsilon' p^t}{(p_0 - p) p'^{t-1}} \right) \\
&\leq \mu_{p'}(\mathcal{F}) \left(\frac{\log_{p'}(\mu_{p'}(\mathcal{F})) + \frac{\epsilon' p^t}{(p_0 - p) p'^{t-1}}}{1 - \epsilon'} \right) \\
&= \mu_{p'}(\mathcal{F}) \left(\log_{p'}(\mu_{p'}(\mathcal{F})) + \frac{\epsilon' \log_{p'}(\mu_{p'}(\mathcal{F})) + \frac{\epsilon' p^t}{(p_0 - p) p'^{t-1}}}{1 - \epsilon'} \right) \\
&\leq \mu_{p'}(\mathcal{F}) \left(\log_{p'}(\mu_{p'}(\mathcal{F})) + 2\epsilon' \log_{p'}(\mu_{p'}(\mathcal{F})) + \frac{2\epsilon' p^t}{(p_0 - p) p'^{t-1}} \right),
\end{aligned} \tag{12}$$

using the fact that $\epsilon' \leq 1/2$.

We claim that

$$\epsilon' \log_{p'}(\mu_{p'}(\mathcal{F})) + \frac{\epsilon' p^t}{(p_0 - p) p'^{t-1}} \leq \frac{3C\epsilon}{\ln\left(\frac{1}{\zeta p_0}\right)}. \tag{13}$$

Indeed, the definition of ϵ' immediately implies that $\frac{\epsilon' p^t}{(p_0 - p) p'^{t-1}} \leq \frac{C\epsilon}{\ln\left(\frac{1}{\zeta p_0}\right)}$. Using the fact that $\mu_p(\mathcal{F}) \geq p^t(1 - \epsilon') \geq p^{t+1}$ and Lemma 2.7(1), we have $\mu_{p'}(\mathcal{F}) \geq p'^{t+1}$, and thus, $\epsilon' \log_{p'}(\mu_{p'}(\mathcal{F})) \leq (t+1)\epsilon' \leq \frac{2C\epsilon}{\ln\left(\frac{1}{\zeta p_0}\right)}$, implying (13).

Combining (12) and (13), we get:

$$p' I_{p'}(\mathcal{F}) \leq \mu_{p'}(\mathcal{F}) \left(\log_{p'}(\mu_{p'}(\mathcal{F})) + \frac{6C\epsilon}{\ln\left(\frac{1}{\zeta p_0}\right)} \right).$$

Therefore, by Theorem 2.4 (and using the assumption $p \geq \zeta p_0$), there exists $B \subset [n]$ such that

$$\mu_{p'}(\mathcal{F} \Delta \mathcal{S}_B) \leq O_\eta \left(\frac{C\epsilon}{\ln\left(\frac{1}{C\epsilon}\right)} \right) \mu_{p'}(\mathcal{F}) \leq p'^t O_\eta \left(\frac{C\epsilon}{\ln\left(\frac{1}{C\epsilon}\right)} \right) \leq p'^t (1 - p') \epsilon \tag{14}$$

(the last inequality using the fact that C is sufficiently small depending on η). Note that $|B| = t$. Indeed, if $|B| \leq t - 1$, then

$$\mu_{p'}(\mathcal{F} \Delta \mathcal{S}_B) \geq \mu_{p'}(\mathcal{S}_B) - \mu_{p'}(\mathcal{F}) \geq (p')^{t-1} - (p')^t = (p')^{t-1}(1 - p') > p'^t (1 - p') \epsilon,$$

contradicting (14). On the other hand, if $|B| \geq t + 1$, then provided C is sufficiently small depending on η , we have $\epsilon' < \frac{1}{2}(1 - p')$, and therefore

$$\begin{aligned}
\mu_{p'}(\mathcal{F} \Delta \mathcal{S}_B) &\geq \mu_{p'}(\mathcal{F}) - \mu_{p'}(\mathcal{S}_B) \geq (p')^t(1 - \epsilon') - (p')^{t+1} = (p')^t(1 - p' - \epsilon') > \frac{1}{2}(p')^t(1 - p') \\
&\geq (p')^t(1 - p')\epsilon,
\end{aligned}$$

again contradicting (14).

Clearly, (14) implies

$$\mu_{p'}(\mathcal{F} \setminus \mathcal{S}_B) \leq p'^t (1 - p') \epsilon. \tag{15}$$

Finally, by Lemma 3.3(a), Equation (15) implies

$$\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq p^{t-1}(1-p)(p'\epsilon)^{\log_{p'} p} \leq p^t \epsilon (1-p) \leq (1-p)p^{t-1} \epsilon.$$

This completes the proof. \square

Remark 3.5. We note that after showing that the total influence $I_{p'}(\mathcal{F})$ is very close to the total influence of a t -umvirate, we can apply Fourier-theoretic tools to show that \mathcal{F} is close to a t -umvirate, instead of using Theorem 2.4. Specifically, for $t = 1$ we can obtain the assertion of Proposition 3.4 using a theorem of Nayar [48], which is a p -biased version of the Friedgut-Kalai-Naor (FKN) theorem [27]. For $t > 1$ one can try to use the higher-degree, p -biased analogue of the FKN theorem due to Kindler and Safra [42] (as in [28]), but this yields a weaker statement than Proposition 3.4. Theorem 2.4 seems to us to be the ‘right’ tool to use in the context.

3.2 A rough stability result for small p

In this subsection we consider the case of a ‘small’ p , i.e., p smaller than some constant depending only on p_0 . We show that for a sufficiently small p , a stability result can be obtained not only when $\mu_p(\mathcal{F})$ is close to p^t , but also under the weaker assumption $\mu_p(\mathcal{F}) \geq Cp^{t+1}$ for some $C = C(p_0, t)$. Our aim is to prove the following.

Proposition 3.6. *For each $p_0 \in (0, 1)$, $t \in \mathbb{N}$ and $\delta > 0$, there exists $C = C(\delta, p_0, t) > 0$ such that the following holds. Let $0 < p < p_0$. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family such that $\mu_{p_0}(\mathcal{F}) \leq p_0^t$, and such that $\mu_p(\mathcal{F}) \geq Cp^{t+1}$. Then there exists $B \in [n]^{(t)}$ such that*

$$\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq \delta^{\log_{(p_0/2)}(p)} p^{t-1}(1-p).$$

For the proof, we introduce a few more definitions.

Definition 3.7. We say that $\mathcal{F} \subset \mathcal{P}([n])$ is *lexicographically ordered* if it is an initial segment of the lexicographic order on $\mathcal{P}([n])$, i.e. whenever $A \in \mathcal{F}$ and $B \subset [n]$ with $\min(A \Delta B) \in B$, we have $B \in \mathcal{F}$. Similarly, we say that $\mathcal{A} \subset [n]^{(k)}$ is *lexicographically ordered* if it is an initial segment of the lexicographic order on $[n]^{(k)}$.

Definition 3.8. If $\mathcal{F} \subset [n]^{(k)}$ and $s \in \mathbb{N}$, the s -upper shadow of \mathcal{F} is defined by

$$\partial^{+(s)}(\mathcal{F}) = \{B \in [n]^{(k+s)} : A \supset B \text{ for some } A \in \mathcal{F}\}.$$

We need the following.

Lemma 3.9. *Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family, let $0 < p < p_0 < 1$, and let $x > 0$. If $\mu_{p_0}(\mathcal{F}) \leq p_0^t(1 - (1 - p_0)^x)$, then $\mu_p(\mathcal{F}) \leq p^t(1 - (1 - p)^x)$.*

Proof. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family such that $\mu_{p_0}(\mathcal{F}) \leq p_0^t(1 - (1 - p_0)^x)$. Let $\epsilon > 0$. Choose $n_0 = n_0(\epsilon) \in \mathbb{N}$ and a lexicographically ordered family $\mathcal{L} = \mathcal{L}(\epsilon) \subset \mathcal{P}([n + n_0])$ such that

$$\mu_{p_0}(\mathcal{F}) < \mu_{p_0}(\mathcal{L}) < \mu_{p_0}(\mathcal{F}) + \epsilon.$$

Note that \mathcal{L} is increasing.

Recall from the proof of Theorem 2.11 that we define

$$\mathcal{F}_{N,k} := \{A \in [N]^{(k)} : A \cap [n] \in \mathcal{F}\},$$

and that for any $\mathcal{F} \subset \mathcal{P}([n])$ and any $p \in (0, 1)$, we have

$$\mu_p(\mathcal{F}) = \lim_{N \rightarrow \infty} \frac{|\mathcal{F}_{N, \lfloor pN \rfloor}|}{\binom{N}{\lfloor pN \rfloor}}. \quad (16)$$

It follows that if N is sufficiently large depending on n , n_0 and p_0 , we have

$$|\mathcal{F}_{N, \lfloor p_0 N \rfloor}| < |\mathcal{L}_{N, \lfloor p_0 N \rfloor}|.$$

Note that for any $N \geq k_0 \geq k \geq n$, and any increasing family $\mathcal{G} \subset \mathcal{P}([n])$ we have

$$\partial^{+(k_0-k)}(\mathcal{G}_{N,k}) = \mathcal{G}_{N,k_0}.$$

Moreover, for any $k \geq n$, $\mathcal{L}_{N,k}$ is lexicographically ordered (as a subset of $[N]^{(k)}$). Now let $k_0 := \lfloor p_0 N \rfloor$, and let $n \leq k \leq k_0$. We claim that $|\mathcal{F}_{N,k}| < |\mathcal{L}_{N,k}|$. Indeed, suppose for a contradiction that $|\mathcal{F}_{N,k}| \geq |\mathcal{L}_{N,k}|$. Then, using the Kruskal-Katona theorem, we have

$$|\mathcal{F}_{N,k_0}| = |\partial^{+(k_0-k)}(\mathcal{F}_{N,k})| \geq |\partial^{+(k_0-k)}(\mathcal{L}_{N,k})| = |\mathcal{L}_{N,k_0}| > |\mathcal{F}_{N,k_0}|,$$

a contradiction. It follows from this claim, and from (16), that

$$\mu_p(\mathcal{F}) \leq \mu_p(\mathcal{L}). \quad (17)$$

Provided ϵ is sufficiently small depending on p_0, t and x , we have

$$p_0^t(1 - (1 - p_0)^x) + \epsilon < p_0^t,$$

and therefore $\mu_{p_0}(\mathcal{L}) < p_0^t$. Hence, we may write $\mu_{p_0}(\mathcal{L}) = p_0^t(1 - (1 - p_0)^y)$, where $y > 0$. Since \mathcal{L} is lexicographically ordered, it is contained in the t -umvirate $\mathcal{S}_{[t]}$, and we have $\mu_{p_0}(\mathcal{L}_{[t]}^{[t]}) = 1 - (1 - p_0)^y$. Hence, by Lemma 2.7, and since $\mathcal{L}_{[t]}^{[t]}$ is increasing, we have $\mu_p(\mathcal{L}_{[t]}^{[t]}) \leq 1 - (1 - p)^y$. Therefore, $\mu_p(\mathcal{L}) \leq p^t(1 - (1 - p)^y)$. Combining this with (17) yields

$$\mu_p(\mathcal{F}) \leq p^t(1 - (1 - p)^y).$$

As $\epsilon \rightarrow 0$, we must have $y \rightarrow x$, so taking the limit of the above as $\epsilon \rightarrow 0$ yields

$$\mu_p(\mathcal{F}) \leq p^t(1 - (1 - p)^x).$$

□

We also need the following immediate corollary of Proposition 3.4.

Corollary 3.10. *Let $0 < p_1 < p_0 < 1$ and let $t \in \mathbb{N}$. For each $\delta > 0$, there exists $x = x(\delta, p_0, p_1, t) > 0$ such that the following holds. If $\mathcal{F} \subset \mathcal{P}([n])$ is an increasing family with $\mu_{p_0}(\mathcal{F}) \leq p_0^t$, and*

$$\mu_{p_1}(\mathcal{F}) \geq p_1^t(1 - (1 - p_1)^x),$$

then there exists $B \in [n]^{(t)}$ such that $\mu_{p_1}(\mathcal{F} \setminus \mathcal{S}_B) \leq \delta p_1^{t-1}(1 - p_1)$.

Proof of Proposition 3.6. Let $p_0, p, t, \delta, \mathcal{F}$ be as in the statement of the proposition. Let $C > 0$ to be specified later. Since $\mu_{p_0}(\mathcal{F}) \leq p_0^t$, we have $\mu_p(\mathcal{F}) \leq p^t$, by Lemma 2.7. Hence, by choosing $C > 2/p_0$, we may assume that $p < p_0/2$. If $Cp \geq 1$, then $\mu_p(\mathcal{F}) \geq p^t$, so $\mu_{p_0}(\mathcal{F}) = p_0^t$, $\mu_p(\mathcal{F}) = p^t$ and \mathcal{F} is a t -umvirate, by Lemma 2.7. Hence, we may assume that $Cp < 1$. Choose $y > 0$ such that $Cp = 1 - (1 - p)^y$; then

$$\mu_p(\mathcal{F}) \geq Cp^{t+1} = p^t (1 - (1 - p)^y).$$

Set $p_1 = p_0/2$. Then $\mu_{p_1}(\mathcal{F}) \geq p_1^t (1 - (1 - p_1)^y)$, by Lemma 3.9. We have

$$y = \log_{1-p}(1 - Cp) = \frac{\ln(1 - Cp)}{\ln(1 - p)} \geq C/2 \quad \forall p \leq 1/2,$$

so provided $C = C(\delta, p_0, t)$ is sufficiently large, we have $y > x(\delta, p_0, p_0/2, t)$ (where $x(\delta, p_0, p_1, t)$ is as in Corollary 3.10), so there exists $B \in [n]^{(t)}$ such that

$$\mu_{p_1}(\mathcal{F} \setminus \mathcal{S}_B) \leq \delta p_1^{t-1} (1 - p_1).$$

It follows from Lemma 3.3 (a) that

$$\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq \delta^{\log_{p_1}(p)} p^{t-1} (1 - p),$$

as required. \square

3.3 A bootstrapping argument

In this subsection we present a bootstrapping argument showing that if an increasing family \mathcal{F} satisfies $\mu_{p_0}(\mathcal{F}) \leq p_0^t$ and is ‘somewhat’ close to a t -umvirate, then it must be ‘very’ close to that t -umvirate.

Proposition 3.11. *Let $t \in \mathbb{N}$, and let $0 < p < p_0 < 1$. Define $\tilde{c} := \left(\frac{1-p_0}{p_0}\right)^{\log_{1-p_0}(1-p)}$ and $u := \log_p(p_0) \cdot \log_{1-p_0}(1-p)$. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family with $\mu_{p_0}(\mathcal{F}) \leq p_0^t$ and with*

$$\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq (1 - p)p^t u^{\frac{1}{1-u}}$$

for some $B \in [n]^{(t)}$. Let $\epsilon > 0$. If

$$\mu_p(\mathcal{F}) \geq p^t \left(1 - \tilde{c}\epsilon^{\log_p(p_0) \cdot \log_{1-p_0}(1-p)}\right) + (1 - p)p^{t-1}\epsilon, \quad (18)$$

then $\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq \epsilon p^{t-1}(1 - p)$.

The proof of Proposition 3.11 has two parts. First, we use an isoperimetric technique encapsulated in Lemma 3.3(b) to obtain an upper bound on $\mu_p(\mathcal{F} \cap \mathcal{S}_B)$ (for some t -umvirate \mathcal{S}_B) in terms of $\mu_p(\mathcal{F} \setminus \mathcal{S}_B)$. (Clearly, the larger $\mu_p(\mathcal{F} \setminus \mathcal{S}_B)$ is, the smaller $\mu_p(\mathcal{F} \cap \mathcal{S}_B)$ can be, but Lemma 3.3(b) yields a sharp bound.) This upper bound yields an upper bound on $\mu_p(\mathcal{F}) = \mu_p(\mathcal{F} \cap \mathcal{S}_B) + \mu_p(\mathcal{F} \setminus \mathcal{S}_B)$ in terms of $\mu_p(\mathcal{F} \setminus \mathcal{S}_B)$. Second, we study the latter bound and show that once $\mu_p(\mathcal{F} \setminus \mathcal{S}_B)$ is assured to be below a certain fixed value, increasing $\mu_p(\mathcal{F} \setminus \mathcal{S}_B)$ only decreases the maximal possible total measure $\mu_p(\mathcal{F})$.

Proof of Proposition 3.11. We may assume w.l.o.g. that $B = [t]$. Let $\delta = \frac{\mu_p(\mathcal{F} \setminus \mathcal{S}_B)}{(1-p)p^{t-1}}$. By Lemma 3.3(b), we have

$$\mu_p(\mathcal{F}) = \mu_p(\mathcal{F} \setminus \mathcal{S}_B) + \mu_p(\mathcal{F} \cap \mathcal{S}_B) \leq (1-p)p^{t-1}\delta + p^t(1 - \tilde{c}\delta^u). \quad (19)$$

Consider the function $f(x) = (1-p)p^{t-1}x + p^t(1 - \tilde{c}x^u)$. It is easy to see that f attains its minimum when $1-p = \tilde{c}pux^{u-1}$, i.e. when

$$\begin{aligned} x &= \left(\frac{\tilde{c}pu}{1-p} \right)^{\frac{1}{1-u}} = u^{\frac{1}{1-u}} \left(\left(\frac{1-p_0}{p_0} \right)^{\log_{1-p_0}(1-p)} \frac{p}{1-p} \right)^{\frac{1}{1-\log_p p_0 \log_{1-p_0}(1-p)}} \\ &= u^{\frac{1}{1-u}} p_0^{\frac{\log_{p_0} p - \log_{1-p_0}(1-p)}{1-\log_p p_0 \log_{1-p_0}(1-p)}} = u^{\frac{1}{1-u}} p_0^{\log_{p_0} p} = u^{\frac{1}{1-u}} p, \end{aligned}$$

and that f is strictly decreasing in the interval $[0, u^{\frac{1}{1-u}}p]$, which contains the point δ . By (18) we have $\mu_p(\mathcal{F}) \geq f(\epsilon)$, while by (19) we have $f(\delta) \geq \mu_p(\mathcal{F})$. Hence, $f(\delta) \geq f(\epsilon)$, and consequently, $\delta \leq \epsilon$, or equivalently, $\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq \epsilon p^{t-1}(1-p)$, as asserted. \square

In order to complete the bootstrapping argument in the case of large p , we will need the following technical claim.

Claim 3.12. *In the notation of Proposition 3.11, let*

$$\nu(p_0, \zeta) := \inf_{p \in [\zeta p_0, p_0]} u^{\frac{1}{1-u}}.$$

Then for any $\eta > 0$, there exists $B = B(\zeta, \eta) > 0$ such that for all $p_0 \leq 1 - \eta$, we have

$$\nu(p_0, \zeta) \geq B.$$

Proof. Since $u \mapsto u^{1/(1-u)}$ is an increasing function of u , and $u = u(p)$ is an increasing function of p for fixed p_0 , it suffices to show that there exists $B = B(\zeta, \eta) > 0$ such that if $p_0 \leq 1 - \eta$, then

$$u(\zeta p_0) = \log_{\zeta p_0}(p_0) \log_{1-p_0}(1 - \zeta p_0) \geq B.$$

This is indeed straightforward. \square

An enhanced bootstrapping result for t -intersecting families

If instead of assuming $\mu_{p_0}(\mathcal{F}) \leq p_0^t$, we assume that \mathcal{F} is t -intersecting, then a stronger bootstrapping result can be obtained by utilising Katona's shadow/intersection theorem (in the form of Theorem 2.11). To show this, we need an analogue of Lemma 3.3.

Lemma 3.13. *Let $t \in \mathbb{N}$ and $0 < p \leq 1/2$. Suppose that $\mathcal{F} \subset \mathcal{P}([n])$ is an increasing t -intersecting family with*

$$\mu_p(\mathcal{F} \setminus \mathcal{S}_{[t]}) \geq (1-p)p^{t-1}\epsilon.$$

Then

$$\mu_p(\mathcal{F} \cap \mathcal{S}_{[t]}) \leq p^t \left(1 - \left(\frac{\epsilon}{2^t - 1} \right)^{\log_p(1-p)} \right). \quad (20)$$

Proof. It is sufficient to prove that for any $\delta > 0$, if $\mu_p(\mathcal{F} \cap \mathcal{S}_{[t]}) \geq p^t(1 - \delta)$, then

$$\mu_p(\mathcal{F} \setminus \mathcal{S}_{[t]}) \leq (2^t - 1) p^{t-1} (1 - p) \delta^{\log_{1-p} p}.$$

We show that for any $B \subsetneq [t]$, we have

$$\mu_p(\mathcal{F}_{[t]}^B) p^{|B|} (1 - p)^{t-|B|} \leq p^{t-1} (1 - p) \delta^{\log_{1-p} p},$$

which clearly implies the assertion.

First, we note that the family $\mathcal{F}_{[t]}^B \subset \mathcal{P}(\{t+1, \dots, n\})$ is $(t - |B|)$ -intersecting (and in particular, $(t - |B| - 1)$ -intersecting). In addition, the families $\partial^{t-|B|-1}(\mathcal{F}_{[t]}^B)$ and $\mathcal{F}_{[t]}^{[t]}$ are cross-intersecting. Hence, we have

$$\frac{(1-p)^{t-|B|-1}}{p^{t-|B|-1}} \mu_p(\mathcal{F}_{[t]}^B) \leq \mu_p(\partial^{t-|B|-1}(\mathcal{F}_{[t]}^B)) \leq \delta^{\log_{1-p} p},$$

where the first inequality uses Theorem 2.11 and the second uses Lemma 2.9. Rearrangement completes the proof. \square

Now we are ready to state the enhanced bootstrapping result.

Proposition 3.14. *Let $t \in \mathbb{N}$ and let $0 < p < 1/(t+1)$. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing, t -intersecting family with $\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq (1-p)p^t(c'v)^{\frac{1}{1-v}}$ for some $B \in [n]^{(t)}$, where $c' := (2^t - 1)^{-\log_p(1-p)}$ and $v := \log_p(1-p)$. Let $\epsilon > 0$. If*

$$\mu_p(\mathcal{F}) \geq p^t \left(1 - \left(\frac{\epsilon}{2^t - 1} \right)^{\log_p 1-p} \right) + (1-p)p^{t-1}\epsilon, \quad (21)$$

then $\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq \epsilon p^{t-1}(1-p)$.

Proof. The proof is essentially the same as the proof of Proposition 3.11 above, with c', v in place of \tilde{c}, u , and Lemma 3.13 in place of Lemma 3.3(b).

Let $\delta = \frac{\mu_p(\mathcal{F} \setminus \mathcal{S}_B)}{(1-p)p^{t-1}}$. By Lemma 3.13, we have

$$\mu_p(\mathcal{F}) = \mu_p(\mathcal{F} \setminus \mathcal{S}_B) + \mu_p(\mathcal{F} \cap \mathcal{S}_B) \leq (1-p)p^{t-1}\delta + p^t(1 - c'\delta^v). \quad (22)$$

Consider the function $g(x) = (1-p)p^{t-1}x + p^t(1 - c'x^v)$. Observe that g attains its minimum when $1-p = c'pvx^{v-1}$, i.e. when

$$x = \left(\frac{c'pv}{1-p} \right)^{\frac{1}{1-v}} = (c'v)^{\frac{1}{1-v}} \left(\frac{p}{1-p} \right)^{\frac{1}{1-v}} = (c'v)^{\frac{1}{1-v}} p,$$

and that g is strictly decreasing in the interval $[0, (c'v)^{\frac{1}{1-v}}p]$, which contains the point δ . By the assumption we have $\mu_p(\mathcal{F}) \geq g(\epsilon)$, while by (22) we have $g(\delta) \geq \mu_p(\mathcal{F})$. Hence, $g(\delta) \geq g(\epsilon)$, and consequently, $\delta \leq \epsilon$, or equivalently, $\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq \epsilon p^{t-1}(1-p)$, as asserted. \square

3.4 Stronger stability results via bootstrapping

In this subsection, we complete the proof of Theorem 3.1.

The case of large p

In the case of large p we prove the following result, which is slightly stronger than the assertion of Theorem 3.1 in the relevant range.

Proposition 3.15. *For any $\zeta, \eta \in (0, 1)$, there exists $c = c(\zeta, \eta) > 0$ such that the following holds. Let $t \in \mathbb{N}$, and let $0 < \zeta p_0 \leq p < p_0 \leq 1 - \eta$. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family with $\mu_{p_0}(\mathcal{F}) \leq p_0^t$ and*

$$\mu_p(\mathcal{F}) \geq p^t \left(1 - \frac{cp_0(p_0 - p)}{\ln(\frac{1}{p_0})t} \right). \quad (23)$$

Let $\epsilon > 0$, and define $\tilde{c} := \left(\frac{1-p_0}{p_0} \right)^{\log_{1-p_0}(1-p)}$. If

$$\mu_p(\mathcal{F}) \geq p^t \left(1 - \tilde{c}\epsilon^{\log_p p_0 \cdot \log_{1-p_0} 1-p} \right) + (1-p)p^{t-1}\epsilon,$$

then there exists a t -umvirate \mathcal{S}_B such that

$$\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq (1-p)p^{t-1}\epsilon.$$

Proof. Denote by m the infimum of the function $p \mapsto pu^{\frac{1}{1-u}}$ in the interval $[\zeta p_0, p_0]$. By Claim 3.12, since $p_0 \leq 1 - \eta$, we have $m \geq Kp_0$, where $K = K(\zeta, \eta) > 0$ depends only on ζ, η . By reducing K if necessary, we may assume that $K \leq 1$. Let \mathcal{F} be a family that satisfies the assumption of Proposition 3.15, with $c = c(\zeta, \eta)$ to be specified later. By Proposition 3.11, in order to show that \mathcal{F} satisfies the conclusion of Proposition 3.15, it suffices to show that there exists a t -umvirate \mathcal{S}_B such that

$$\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq (1-p)p^{t-1}Kp_0.$$

And indeed, if $\mu_p(\mathcal{F}) \geq p^t(1 - \frac{cp_0(p_0-p)}{\ln(\frac{1}{p_0})t})$, where $c = c(\zeta, \eta) > 0$ is sufficiently small, then $\mu_p(\mathcal{F}) \geq p^t(1 - \epsilon')$, where $\epsilon' = \frac{C(\eta)Kp_0(p_0-p)}{\ln(\frac{1}{\zeta p_0})t}$, $C(\eta)$ being the constant of Proposition 3.4. Hence, by Proposition 3.4 (applied with $\epsilon = Kp_0$), we have $\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq (1-p)p^{t-1}Kp_0$. This completes the proof. \square

The case of small p

For the case of small p , we prove the following.

Proposition 3.16. *For any $p_0 \in (0, 1)$ and $t \in \mathbb{N}$, there exists $C = C(p_0, t) > 0$ such that the following holds. Let $0 < p < p_0$, and let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family with $\mu_{p_0}(\mathcal{F}) \leq p_0^t$ and $\mu_p(\mathcal{F}) \geq Cp^{t+1}$. Let $\epsilon > 0$, and define $\tilde{c} := \left(\frac{1-p_0}{p_0} \right)^{\log_{1-p_0}(1-p)}$. If*

$$\mu_p(\mathcal{F}) \geq p^t \left(1 - \tilde{c}\epsilon^{\log_p p_0 \cdot \log_{1-p_0} 1-p} \right) + p^{t-1}(1-p)\epsilon,$$

then there exists a t -umvirate \mathcal{S}_B such that $\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq \epsilon p^{t-1}(1-p)$.

Proof. Let $\mathcal{F} \subset \mathcal{P}([n])$ be a family that satisfies the assumption of the proposition. By Lemma 2.7 (1), we have $\mu_p(\mathcal{F}) \leq p^t$. Let $\zeta = \zeta(p_0, t) > 0$. If we choose $C > 1/\zeta$, then $Cp^{t+1} > p^t$ for any $p \geq \zeta$, so we may assume throughout that $p < \zeta$.

We claim that the assertion of the proposition now follows from Propositions 3.6 and 3.11. Let $\delta = \delta(p_0, t) > 0$ to be chosen later. By Proposition 3.6, provided $C = C(p_0, t)$ is sufficiently large, there exists $B \in [n]^{(t)}$ such that

$$\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq \delta^{\log_{(p_0/2)}(p)} p^{t-1} (1-p).$$

Proposition 3.16 will follow from Proposition 3.11 once we have shown that

$$\delta^{\log_{(p_0/2)}(p)} \leq pu^{\frac{1}{1-u}}.$$

We have

$$pu^{\frac{1}{1-u}} = \Theta_{p_0} \left((p \log_p p_0 \log_{1-p_0} (1-p))^{1+o(1)} \right) = \Theta_{p_0} (p^{2+o(1)}),$$

(where $o(1)$ denotes a function of p, p_0 which tends to zero as $p \rightarrow 0$ for any fixed $p_0 \in (0, 1)$). Hence, it suffices to prove that

$$\delta^{\log_{(p_0/2)}(p)} \leq \eta_{p_0} p^{2+o(1)}$$

(where $\eta_{p_0} > 0$ depends only on p_0), i.e., that

$$\frac{\ln(1/p) \ln(1/\delta)}{\ln(2/p_0)} \geq \ln(1/\eta_{p_0}) + (2 + o(1)) \ln(1/p).$$

This holds provided we choose $\delta < (p_0/2)^3$ and provided p is sufficiently small depending on p_0 and t . Therefore, Proposition 3.11 can be applied to \mathcal{F} to yield the assertion of Proposition 3.16. \square

Wrapping up the proof of Theorem 3.1

Theorem 3.1 follows quickly from Propositions 3.15 and 3.16. Indeed, let $C = C(p_0, t)$ be the constant in Proposition 3.16. Choose $\zeta = \zeta(p_0, t) > 0$ such that $C\zeta p_0 < 1$. Apply Proposition 3.16 for all $p \in (0, \zeta p_0)$. Now apply Proposition 3.15 with $\eta = 1 - p_0$ and $\zeta = \zeta(p_0, t)$, for all $p \in [\zeta p_0, p_0)$, yielding $c = c(p_0, t) > 0$ such that for all $p \in [\zeta p_0, p_0)$, the conclusion of Proposition 3.15 holds when the condition (23) is replaced by the condition

$$\mu_p(\mathcal{F}) \geq p^t (1 - c(p_0 - p)).$$

Now increase the value of C or reduce the value of c if necessary so as to ensure that $C\zeta p_0 = 1 - c(p_0 - \zeta p_0)$. Then

$$\min\{Cp^{t+1}, p^t(1 - c(p_0 - p))\} = \begin{cases} Cp^{t+1} & \text{if } 0 < p \leq \zeta p_0; \\ p^t(1 - c(p_0 - p)) & \text{if } \zeta p_0 \leq p < p_0. \end{cases}$$

This completes the proof of Theorem 3.1.

3.5 t -Intersecting families – a stability result for the biased Wilson theorem

By the biased Wilson theorem (Theorem 1.7), if $\mathcal{F} \subset \mathcal{P}([n])$ is a t -intersecting family, then $\mu_{t/(t+1)}(\mathcal{F}) \leq (t+1)^{-t}$. Hence, a direct application of Theorem 3.1 with $p_0 = 1/(t+1)$ already yields a rather strong stability version of the biased Wilson theorem. However, for $t > 1$, the ϵ -dependence in Theorem 3.1 is not sharp for t -intersecting families. (The tightness example $\mathcal{H}_{t,s,r}$ for Theorem 3.1 is not t -intersecting for $t > 1$ and $r > s$, and when $t > 1$, the condition $(1 - p_0)^{r-1} = p_0^{s-1}$ implies $r > s$.) By utilising the t -intersection condition, we obtain the following stability version of the biased Wilson theorem.

Theorem 3.17. *For any $t \in \mathbb{N}$, there exist $C = C(t)$, $c = c(t) > 0$ such that the following holds. Let $0 < p < 1/(t+1)$, and let $\mathcal{F} \subset \mathcal{P}([n])$ be a t -intersecting family such that*

$$\mu_p(\mathcal{F}) \geq \min\{Cp^{t+1}, p^t(1 - c(\frac{1}{t+1} - p))\}. \quad (24)$$

Let $\epsilon > 0$. If

$$\mu_p(\mathcal{F}) \geq p^t \left(1 - \left(\frac{\epsilon}{2^t - 1} \right)^{\log_p(1-p)} \right) + (1-p)p^{t-1}\epsilon, \quad (25)$$

then $\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq (1-p)p^{t-1}\epsilon$ for some t -umvirate \mathcal{S}_B .

The ϵ -dependence in Theorem 3.17 is sharp, up to a factor depending only on t , as evidenced by the families $\{\tilde{\mathcal{F}}_{t,s}\}_{t,s \in \mathbb{N}}$, defined as follows.

$$\begin{aligned} \tilde{\mathcal{F}}_{t,s} := & \{A \subset \mathcal{P}([n]) : [t] \subset A, \{t+1, \dots, t+s\} \cap A \neq \emptyset\} \\ & \cup \{A \subset \mathcal{P}([n]) : |[t] \cap A| = t-1, \{t+1, \dots, t+s\} \subset A\}. \end{aligned}$$

Indeed, we have

$$\mu_p(\tilde{\mathcal{F}}_{t,s}) = p^t(1 - (1-p)^s) + tp^{t-1}(1-p)p^s$$

and

$$\mu_p(\tilde{\mathcal{F}}_{t,s} \setminus \mathcal{S}_{[t]}) = tp^{t-1}(1-p)p^s,$$

which corresponds to the assertion of Theorem 3.17 with $\epsilon = tp^s$, provided we replace $2^t - 1$ by t in the condition (25). (Note that the condition (24) is satisfied provided s is sufficiently large.) In fact, we conjecture that the families $\tilde{\mathcal{F}}_{t,s}$ are precisely extremal (that is, the factor of $2^t - 1$ in the statement of the theorem could be replaced by t); see Section 6.

The proof of Theorem 3.17 is almost exactly the same as the proof of Theorem 3.1, except that the ‘enhanced’ bootstrapping result for t -intersecting families (Proposition 3.14) is used in place of the bootstrapping result for arbitrary increasing families (Proposition 3.11); we omit the details.

3.6 A stronger stability result for the Simonovits-Sós conjecture.

Let \mathcal{F} be a family of labelled graphs with vertex-set $[n]$, i.e. $\mathcal{F} \subset \mathcal{P}([n]^{(2)})$. The family \mathcal{F} is said to be *triangle-intersecting* if any two graphs in \mathcal{F} share some triangle. A well-known conjecture of Simonovits and Sós from 1976 asserted that if $\mathcal{F} \subset \mathcal{P}([n]^{(2)})$ is triangle-intersecting, then $|\mathcal{F}| \leq \frac{1}{8}2^{\binom{n}{2}}$. In 2012, Ellis, Filmus and Friedgut [18] proved this conjecture, and in fact proved the slightly stronger statement that any triangle-intersecting family $\mathcal{F} \subset \mathcal{P}([n]^{(2)})$ satisfies $\mu_p(\mathcal{F}) \leq p^3$ for all $p \leq 1/2$. Furthermore, they proved in [18] a stability version stating that if a triangle-intersecting $\mathcal{F} \subset \mathcal{P}([n]^{(2)})$ satisfies $\mu_p(\mathcal{F}) \geq (1-\epsilon)p^3$, then there exists a triangle T such that $\mu_p(\mathcal{F} \setminus \mathcal{S}_T) \leq c\epsilon$, where $\mathcal{S}_T := \{G \subset [n]^{(2)} : T \subset G\}$ and c is an absolute constant.

Theorem 3.1 (applied for $t = 3$ and $p_0 = 1/2$), along with the result of [18] that any triangle-intersecting family \mathcal{F} satisfies $\mu_{1/2}(\mathcal{F}) \leq 1/8$, yields the following stability result.

Corollary 3.18. *There exist absolute constants $C, c > 0$ such that the following holds. Let $0 < p < 1/2$ and let $\mathcal{F} \subset \mathcal{P}([n]^{(2)})$ be a triangle-intersecting family with*

$$\mu_p(\mathcal{F}) \geq \min\{Cp^4, p^3(1 - c(\frac{1}{2} - p))\}.$$

Let $\epsilon > 0$. If

$$\mu_p(\mathcal{F}) \geq p^3 \left(1 - \epsilon^{\log_p(1-p)}\right) + p^2(1-p)\epsilon,$$

then $\mu_p(\mathcal{F} \setminus \mathcal{S}_T) \leq (1-p)p^2\epsilon$ for some triangle T .

Corollary 3.18 is much stronger than the stability result in [18] in the case where p is bounded away from $1/2$. For example, if $\mu_p(\mathcal{F}) = (1-\delta)p^3$ and $p = 1/4$, then it yields $\mu_{1/4}(\mathcal{F} \setminus \mathcal{S}_T) = O(\delta^{\log_p(1-p)}) = O(\delta^{4.8})$, compared to $\mu_{1/4}(\mathcal{F} \setminus \mathcal{S}_T) = O(\delta)$ from the stability result of [18]. We believe, however, that Corollary 3.18 is not tight in its ϵ -dependence, and that to obtain a tight result one would have to exploit in a more significant way the fact that the family is triangle-intersecting (not just 3-intersecting); see Section 6.

3.7 The ‘dual’ stability theorem and an application to the Erdős matching conjecture

We now give our ‘dual’ version of Theorem 3.1, which allows us to obtain stability results for EKR-type theorems in which the extremal example is the *dual* of a t -umvirate.

Theorem 3.19. *Let $s \in \mathbb{N}$, and let $0 < p_0 < 1$. Then there exist $C = C(p_0, s)$, $c = c(p_0, s) > 0$ such that the following holds. Let $0 < p < p_0$, and let $\mathcal{F} \subset \mathcal{P}([n])$ be an increasing family such that $\mu_{p_0}(\mathcal{F}) \leq 1 - (1 - p_0)^s$ and*

$$\mu_p(\mathcal{F}) \geq \min\{(s-1)p + Cp^2, (1 - c(p_0 - p))(1 - (1-p)^s)\}. \quad (26)$$

Let $\epsilon > 0$, and define $\tilde{c} := \left(\frac{1-p_0}{p_0}\right)^{\log_{1-p_0}(1-p)}$. If

$$\mu_p(\mathcal{F}) \geq 1 - (1-p)^{s-1} + (1-p)^{s-1} \left(p \left(1 - \tilde{c} \epsilon^{\log_p p_0 \log_{1-p_0}(1-p)} \right) + (1-p)\epsilon \right), \quad (27)$$

then there exists $B \in [n]^{(s)}$ such that

$$\mu_p(\mathcal{F} \setminus \text{OR}_B) \leq (1-p)^s \epsilon. \quad (28)$$

Remark 3.20. For any $\xi \in (0, 1/2]$, we have

$$\sup_{p_0 \in [\xi, 1-\xi]} C(p_0, s) = O_{\xi, s}(1), \quad \inf_{p_0 \in [\xi, 1-\xi]} c(p_0, s) = \Omega_{\xi, s}(1).$$

Theorem 3.19 is tight for infinitely many of the families $\{\tilde{\mathcal{D}}_{s,d,l} : s, d, l \in \mathbb{N}\}$, defined by:

$$\begin{aligned} \tilde{\mathcal{D}}_{s,d,l} = & \{A \subset [n] : A \cap [s-1] \neq \emptyset\} \\ & \cup \{A \subset [n] : (A \cap [s] = \{s\}) \wedge (A \cap \{s+1, s+2, \dots, s+d\} \neq \emptyset)\} \\ & \cup \{A \subset [n] : (A \cap [s] = \emptyset) \wedge (\{s+1, s+2, \dots, s+l\} \subset A)\}. \end{aligned}$$

To see this, for each $(s, d, l) \in \mathbb{N}^3$ with $d, l \geq 2$, choose the unique $p_0 \in (0, 1)$ such that $(1-p_0)^{d-1} = p_0^{l-1}$. Then we have $\mu_{p_0}(\mathcal{D}_{s,d,l}) = 1 - (1-p_0)^s$, $\mu_p(\mathcal{D}_{s,d,l}) = 1 - (1-p)^{s-1} + (1-p)^{s-1}(p(1 - (1-p)^d) + (1-p)p^l)$, and $\mu_p(\mathcal{D}_{s,d,l} \setminus \text{OR}_{[s]}) = (1-p)^s p^l$, so equality holds in (27) and (28) with $\epsilon = p^l$. Provided d is sufficiently large, the condition (26) is also satisfied.

The proof of Theorem 3.19 is very similar indeed to the proof of Theorem 3.1 (taking the dual of \mathcal{F} in the appropriate places), so is omitted.

Application to the Erdős matching conjecture

For $\mathcal{F} \subset \mathcal{P}([n])$, the *matching number* $m(\mathcal{F})$ of \mathcal{F} is the maximum integer s such that \mathcal{F} contains s pairwise disjoint sets. The well-known 1965 *Erdős matching conjecture* [20] asserts that if $n, k, s \in \mathbb{N}$ with $n \geq (s+1)k$ and $\mathcal{F} \subset [n]^{(k)}$ with $m(\mathcal{F}) \leq s$, then

$$|\mathcal{F}| \leq \max \left\{ \binom{n}{k} - \binom{n-s}{k}, \binom{k(s+1)-1}{k} \right\}.$$

This conjecture remains open. Erdős himself proved the conjecture for all n sufficiently large, i.e. for $n \geq n_0(k, s)$. The bound on $n_0(k, s)$ was lowered in several works: Bollobás, Daykin and Erdős [9] showed that $n_0(k, s) \leq 2sk^3$; Huang, Loh and Sudakov [33] showed that $n_0(k, s) \leq 3sk^2$, and Frankl and Füredi (unpublished) showed that $n_0(k, s) \leq cks^2$. The most significant result to date is the following theorem of Frankl [24]:

Theorem 3.21 (Frankl, 2013). *Let $n, k, s \in \mathbb{N}$ such that $n > (2s+1)k - s$, and let $\mathcal{F} \subset [n]^{(k)}$ such that $m(\mathcal{F}) \leq s$. Then $|\mathcal{F}| \leq \binom{n}{k} - \binom{n-s}{k}$. Equality holds if and only if there exists $B \in [n]^{(s)}$ such that $\mathcal{F} = \{A \in [n]^{(k)} : A \cap B \neq \emptyset\}$.*

Frankl's Theorem immediately implies the following, via the method of 'going to infinity and back'.

Corollary 3.22. *Let $n, s \in \mathbb{N}$, and let $p \leq 1/(2s+1)$. Let $\mathcal{F} \subset \mathcal{P}([n])$ such that $m(\mathcal{F}) \leq s$. Then $\mu_p(\mathcal{F}) \leq 1 - (1-p)^s$.*

Using the $p = \frac{1}{2s+1}$ case of Corollary 3.22, we may apply Theorem 3.19 (with $p_0 = \frac{1}{2s+1}$) to immediately yield the following stability version of Corollary 3.22. (Note that in the proof, we may assume w.l.o.g. that \mathcal{F} is increasing.)

Corollary 3.23. *For any $s \in \mathbb{N}$, there exist $C = C(s)$, $c = c(s) > 0$ such that the following holds. Let $0 < p < \frac{1}{2s+1}$, and let $\mathcal{F} \subset \mathcal{P}([n])$ such that $m(\mathcal{F}) \leq s$ and*

$$\mu_p(\mathcal{F}) \geq \min \left\{ (s-1)p + Cp^2, (1 - c(\frac{1}{2s+1} - p))(1 - (1-p)^s) \right\}.$$

Let $\epsilon > 0$, and define $\tilde{c} := (2s)^{\log_{2s/(2s+1)}(1-p)}$. If

$$\mu_p(\mathcal{F}) \geq 1 - (1-p)^s - (1-p)^{s-1} p \tilde{c} \epsilon^{\log_p(1/(2s+1)) \log_{2s/(2s+1)}(1-p)} + (1-p)^s \epsilon,$$

then there exists $B \in [n]^{(s)}$ such that

$$\mu_p(\mathcal{F} \setminus \text{OR}_B) \leq (1-p)^s \epsilon.$$

3.8 Additional applications

It is easy to show that Theorem 3.1 can be used to obtain a variety of other stability results for EKR-type theorems. For example, it implies directly (a stronger version of) all results of [35] and the main result of [53], as well as stability versions of all currently known exact results on r -wise (cross)- t -intersecting families (see, e.g., [25, 52, 53] and the references therein). As these derivations are straightforward, we do not present them in this paper.

4 Families of k -element sets

In this section we leverage our main results from the biased-measure setting to the more classical setting of subfamilies of $[n]^{(k)}$, often called the ‘ k -uniform’ setting. Most of the section is devoted to the proof of Theorem 1.5, i.e., a stability result for Wilson’s theorem. After that, we present the k -uniform versions of our stability results for the Simonovits-Sós conjecture and the Erdős matching conjecture.

Throughout this section, if $\mathcal{F} \subset \mathcal{P}([n])$ and $k \in [n]$, we will often write $\mathcal{F}^{(k)} := \mathcal{F} \cap [n]^{(k)}$ for brevity, abusing notation slightly.

We will make repeated use of the following simple Chernoff bound (see, e.g., [4], Appendix A).

Proposition 4.1. *Let X be a random variable with $X \sim \text{Bin}(n, p)$, and let $\delta \in [0, 1]$. Then*

$$\Pr\{X \geq (1 + \delta)np\} < e^{-\delta^2 np/3}, \quad \Pr\{X \leq (1 - \delta)np\} < e^{-\delta^2 np/2}.$$

4.1 A stability result for Wilson’s theorem

In this section we prove Theorem 1.5, our almost-sharp stability result for Wilson’s theorem (Theorem 1.3), improving the stability result of Friedgut in [28]. Let us recall the statement of Theorem 1.5.

Theorem. *For any $t \in \mathbb{N}$ and $\eta > 0$, there exists $\delta_0 = \delta_0(\eta, t) > 0$ such that the following holds. Let $k, n \in \mathbb{N}$ with $\frac{k}{n} \leq \frac{1}{t+1} - \eta$, and let $d \in \mathbb{N}$. Let $\mathcal{A} \subset [n]^{(k)}$ be a t -intersecting family with*

$$|\mathcal{A}| > \max \left\{ \binom{n-t}{k-t} (1 - \delta_0), \binom{n-t}{k-t} - \binom{n-t-d}{k-t} + (2^t - 1) \binom{n-t-d}{k-t-d+1} \right\}.$$

Then there exists a t -umvirate \mathcal{S}_B such that

$$|\mathcal{A} \setminus \mathcal{S}_B| \leq (2^t - 1) \binom{n-t-d}{k-t-d+1}.$$

As in the proof of Theorem 3.1, we first prove a weak stability result, and then we prove a bootstrapping lemma that allows us to leverage our weak stability result into a stronger stability result.

4.1.1 A weak stability result

We start with two lemmas.

Lemma 4.2. *Let $n, k, l \in \mathbb{N}$ with $n \geq k + l$, let $r \in \mathbb{N} \cup \{0\}$, and let $\mathcal{A} \subset [n]^{(k)}$, $\mathcal{B} \subset [n]^{(l)}$ be cross-intersecting families. Suppose that $|\mathcal{A}| \geq \binom{n}{k} - \binom{n-r}{k}$. Then $|\mathcal{B}| \leq \binom{n-r}{l}$.*

Proof. Let $\overline{\mathcal{A}} := \{[n] \setminus A : A \in \mathcal{A}\}$; then $|\overline{\mathcal{A}}| \geq \binom{n}{n-k} - \binom{n-r}{n-k-r}$. By the Kruskal-Katona theorem, we have $|\partial^{n-k-l}(\overline{\mathcal{A}})| \geq \binom{n}{l} - \binom{n-r}{l-r}$. Since \mathcal{A} and \mathcal{B} are cross-intersecting, we have $\mathcal{B} \cap \partial^{n-k-l}(\overline{\mathcal{A}}) = \emptyset$. Hence, $|\mathcal{B}| \leq \binom{n}{l} - |\partial^{n-k-l}(\overline{\mathcal{A}})| \leq \binom{n-r}{l-r}$, as required. \square

Combining this with Theorem 2.10 yields the following.

Lemma 4.3. *Let $n, k, l, t \in \mathbb{N}$ with $n \geq k + l - 2t + 1$, let $r \in \mathbb{N} \cup \{0\}$, and let $B \in [n]^{(t)}$. Let $\mathcal{F} \in \mathcal{P}([n])$ be a t -intersecting family such that*

$$|\mathcal{F}^{(k)} \cap \mathcal{S}_B| \geq \binom{n-t}{k-t} - \binom{n-t-r}{k-t}.$$

Then $|\mathcal{F}^{(l)} \setminus \mathcal{S}_B| \leq (2^t - 1) \binom{n-t-r}{l-t-r+1}$.

Proof. Without loss of generality, we may assume that $B = [t]$. It suffices to show that for each $C \subsetneq [t]$, we have $|\mathcal{F}_{[t]}^C|^{(l-|C|)}| \leq \binom{n-t-r}{l-t-r+1}$.

Note that $\mathcal{F}_{[t]}^C$ is $(t-|C|)$ -intersecting (and in particular, $(t-|C|-1)$ -intersecting), and that $\partial^{t-|C|-1}(\mathcal{F}_{[t]}^C)$ and $\mathcal{F}_{[t]}^{[t]}$ are cross-intersecting. Applying Theorem 2.10 to $\mathcal{F}_{[t]}^C$ and Lemma 4.2 to the pair $(\mathcal{F}_{[t]}^{[t]})^{(k-t)}$, $\partial^{t-|C|-1}(\mathcal{F}_{[t]}^C)$, we get

$$|\mathcal{F}_{[t]}^C|^{(l-|C|)}| \leq \left| \partial^{t-|C|-1} \left((\mathcal{F}_{[t]}^C)^{(l-|C|)} \right) \right| \leq \binom{n-t-r}{l-t-r+1},$$

as asserted. \square

This, together with our results for the p -biased measure on $\mathcal{P}([n])$, enables us to prove the following weak stability result.

Proposition 4.4. *Let $t, n \in \mathbb{N}$, let $0 < \eta < \frac{1}{t+1}$, let $\epsilon > 0$, and let $k \in \mathbb{N}$ be such that $k \leq (\frac{1}{t+1} - \eta)n$. Then there exists $\delta = \delta(\epsilon, \eta, t) > 0$ such that the following holds. Let $\mathcal{A} \subset [n]^{(k)}$ be a t -intersecting family with*

$$|\mathcal{A}| \geq (1 - \delta) \binom{n-t}{k-t}.$$

Then there exists $B \in [n]^{(t)}$ such that

$$|\mathcal{A} \setminus \mathcal{S}_B| \leq \epsilon \binom{n-t}{k-t}.$$

Proof. By the equality case of Theorem 1.3, by making $\delta = \delta(\epsilon, \eta, t) > 0$ smaller if necessary, we may assume throughout that $n \geq n_0(\epsilon, \eta, t)$ for any $n_0(\epsilon, \eta, t) \in \mathbb{N}$.

Choose $\delta > 0$ sufficiently small such that

$$|\mathcal{A}| \geq \binom{n-t}{k-t} - \binom{n-t-s}{k-t},$$

where $s = s(\epsilon, \eta, t) \in \mathbb{N}$ will be chosen later. Let $\mathcal{F} = \mathcal{A}^\uparrow$ be the minimal increasing subfamily of $\mathcal{P}([n])$ that contains \mathcal{A} . Define

$$\mathcal{C}_{t,s} := \{F \subset [n] : [t] \subset F, F \cap \{t+1, \dots, t+s\} \neq \emptyset\};$$

note that $\mathcal{C}_{t,s}^{(l)}$ is an initial segment of the lexicographic ordering on $[n]^{(l)}$, for any $t+1 \leq l \leq n$. We have

$$|\mathcal{F}^{(k)}| = |\mathcal{A}| \geq \binom{n-t}{k-t} - \binom{n-t-s}{k-t} = |(\mathcal{C}_{t,s})^{(k)}|,$$

so by the Kruskal-Katona theorem, we have

$$|\mathcal{F}^{(l)}| \geq |(\mathcal{C}_{t,s})^{(l)}| = \binom{n-t}{l-t} - \binom{n-t-s}{l-t} \quad \forall l \geq k.$$

Define $p_0 := 1/(t+1)$, $p_1 := k/n$, $p := (p_0 + p_1)/2$. By the Chernoff bound in Proposition 4.1, we have

$$\mu_p(\{F \subset [n] : |F| < k\}) \leq \Pr\{\text{Bin}(n, p) < (1-\eta)pn\} < e^{-\eta^2 pn/2} \leq e^{-\eta^2 n/(4(t+1))} = o_{\eta,t}(1),$$

where $o_{\eta,t}(1)$ denotes a function of n tending to 0 as $n \rightarrow \infty$, for fixed η, t . Hence,

$$\begin{aligned} \mu_p(\mathcal{F}) &\geq \mu_p(\{F \subset [n] : |F| \geq k\}) \mu_p(\{F \in \mathcal{F} : |F| \geq k\}) \\ &\geq (1 - o_{\eta,t}(1)) \mu_p(\mathcal{C}_{t,s}) = (1 - o_{\eta,t}(1)) p^t (1 - (1-p)^s). \end{aligned}$$

On the other hand, since \mathcal{F} is t -intersecting, we have $\mu_{p_0}(\mathcal{F}) \leq p_0^t$. Therefore, by Theorem 3.17, provided

$$(1-p)^s + o_{\eta,t}(1) \leq c\left(\frac{1}{t+1} - p\right), \quad (29)$$

there exists $B \in [n]^{(t)}$ such that

$$\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq (1-p)p^{t-1}\epsilon_1,$$

where ϵ_1 is the smallest positive solution to

$$\left(\frac{\epsilon_1}{2^t - 1}\right)^{\log_p(1-p)} - \frac{1-p}{p}\epsilon_1 = (1-p)^s + o_{\eta,t}(1).$$

Using the fact that $p \in [1/(2(t+1)), 1/(t+1) - \eta/2]$, provided s, n are sufficiently large depending on η, t , condition (29) holds and we have $\epsilon_1 \leq O_t(1)p^s + o_{\eta,t}(1)$. Hence,

$$\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq p^t(O_t(1)p^s + o_{\eta,t}(1)),$$

and therefore

$$\mu_p(\mathcal{F} \cap \mathcal{S}_B) \geq p^t(1 - (1-p)^s - O_t(1)p^s - o_{\eta,t}(1)). \quad (30)$$

Now choose $r = r(\epsilon, t, \eta) \in \mathbb{N}$ such that

$$(2^t - 1) \binom{n-t-r}{k-t-r+1} \leq \epsilon \binom{n-t}{k-t}.$$

We claim that

$$|\mathcal{A} \setminus \mathcal{S}_B| \leq (2^t - 1) \binom{n-t-r}{k-t-r+1}, \quad (31)$$

which implies the assertion of the proposition.

Suppose on the contrary that (31) fails. Then by Lemma 4.3, we have

$$|\mathcal{F}^{(l)} \cap \mathcal{S}_B| < \binom{n-t}{l-t} - \binom{n-t-r}{l-t} = |(\mathcal{C}_{t,r})^{(l)}| \quad \forall l \leq n - k + 2t - 1.$$

Using the Chernoff bound in Proposition 4.1 as above, it is easy to see that

$$\mu_p(\{F \subset [n] : |F| \leq n - k + 2t - 1\}) = 1 - o_{\eta,t}(1).$$

Thus,

$$\begin{aligned}\mu_p(\mathcal{F} \cap \mathcal{S}_B) &\leq \mu_p(\{|F| \leq n - k + 2t - 1\})\mu_p(\{F \in \mathcal{F} \cap \mathcal{S}_B \mid \{|F| \leq n - k + 2t - 1\}\}) + o_{\eta,t}(1) \\ &\leq (1 - o_{\eta,t}(1))\mu_p(\mathcal{C}_{t,r}) + o_{\eta,t}(1) = (1 - o_{\eta,t}(1))p^t(1 - (1 - p)^r) + o_{\eta,t}(1) \\ &= p^t(1 - (1 - p)^r) + o_{\eta,t}(1),\end{aligned}$$

contradicting (30) if s, n are sufficiently large depending on η, t and ϵ . This completes the proof. \square

4.1.2 A bootstrapping argument

First, we cite an old result of Hilton (see [23], Theorem 1.2).

Notation 4.5. For $X \subset \mathbb{N}$, $i \in \mathbb{N}$ and $\mathcal{A} \subset X^{(i)}$, we write $\mathcal{L}(\mathcal{A})$ for the initial segment of the lexicographic order on $X^{(i)}$ with size $|\mathcal{A}|$. We say a family $\mathcal{C} \subset X^{(i)}$ is *lexicographically ordered* if it is an initial segment of the lexicographic order on $X^{(i)}$, i.e., $\mathcal{L}(\mathcal{C}) = \mathcal{C}$.

Proposition 4.6 (Hilton). *If $\mathcal{A} \subset [n]^{(k)}$, $\mathcal{B} \subset [n]^{(l)}$ are cross-intersecting, then $\mathcal{L}(\mathcal{A})$, $\mathcal{L}(\mathcal{B})$ are also cross-intersecting.*

We use the following technical lemma.

Lemma 4.7. *For any $\eta > 0$ and any $C \geq 0$, there exists $c = c_0(\eta, C) \in \mathbb{N}$ such that the following holds. Let $n, l, k, d \in \mathbb{N} \cup \{0\}$ with $n \geq (1 + \eta)l + k + c$ and $l \geq k + c - 1$. Suppose that $\mathcal{A} \subset [n]^{(l)}$, $\mathcal{B} \subset [n]^{(k)}$ are cross-intersecting, and that*

$$|\mathcal{A}| \leq |\text{OR}_{[d]} \cap [n]^{(l)}| = \binom{n}{l} - \binom{n-d}{l}.$$

Then

$$|\mathcal{A}| + C|\mathcal{B}| \leq \binom{n}{l} - \binom{n-d}{l} + C\binom{n-d}{k-d}.$$

Proof. We prove the lemma by induction on k . For $k = 0$ the lemma holds trivially. Assume now that $k \geq 1$, and that the statement of the lemma holds for $k - 1$. For $d = 0$, the statement of the lemma holds trivially, so we may assume throughout that $d \geq 1$. By Proposition 4.6, we may assume that \mathcal{A} and \mathcal{B} are lexicographically ordered. Since $d \geq 1$, we have $|\mathcal{A}| \leq \binom{n}{l} - \binom{n-1}{l} = \binom{n-1}{l-1}$, so $\mathcal{A} \subset \mathcal{F}_1^{(l)}$, where $\mathcal{F}_1^{(i)} := \{A \in [n]^{(i)} : 1 \in A\}$ for each $i \in [n]$.

We split into two cases: $\mathcal{A} = \mathcal{F}_1^{(l)}$ and $\mathcal{A} \subsetneq \mathcal{F}_1^{(l)}$.

Case 1: $\mathcal{A} = \mathcal{F}_1^{(l)}$. First note that $\mathcal{B} \subset \mathcal{F}_1^{(k)}$. Indeed, suppose on the contrary that $B \in \mathcal{B}$ and $1 \notin B$. Since $n \geq k + l$, there exists $A \in [n]^{(l)}$ such that $1 \in A$ and $A \cap B = \emptyset$. Hence, $A \in \mathcal{F}_1^{(l)} = \mathcal{A}$, and $A \cap B = \emptyset$, a contradiction. Hence, we may assume that $\mathcal{B} = \mathcal{F}_1^{(k)}$. We must prove that

$$\binom{n-1}{l-1} + C\binom{n-1}{k-1} \leq \binom{n}{l} - \binom{n-d}{l} + C\binom{n-d}{k-d} \quad \forall d \geq 1. \quad (32)$$

This clearly holds (with equality) if $d = 1$. To verify it for all $d \geq 2$ it suffices to show that

$$\binom{n-1}{l-1} + C\binom{n-1}{k-1} \leq \binom{n}{l} - \binom{n-2}{l},$$

or equivalently,

$$C \binom{n-1}{k-1} \leq \binom{n-2}{l-1}.$$

We have

$$\begin{aligned} \frac{\binom{n-1}{k-1}}{\binom{n-2}{l-1}} &= \frac{n-1}{n-k} \frac{\binom{n-2}{k-1}}{\binom{n-2}{l-1}} \leq 2 \frac{(l-1)(l-2) \dots k}{(n-k-1)(n-k-2) \dots (n-l)} \\ &\leq 2 \left(\frac{l-1}{n-k-1} \right)^{l-k} \leq 2 \left(\frac{l-2}{l+\eta l+c-1} \right)^{c-1} \leq \frac{1}{C}, \end{aligned}$$

provided c is sufficiently large depending on η and C , as required.

Case 2: $\mathcal{A} \subsetneq \mathcal{F}_1^{(l)}$. If $|\mathcal{A}| \leq \binom{n-2}{l-2}$, then

$$|\mathcal{A}| + C |\mathcal{B}| \leq \binom{n-2}{l-2} + C \binom{n}{k} \leq \binom{n-1}{l-1} \leq \binom{n}{l} - \binom{n-d}{l} + C \binom{n-d}{k-d},$$

where the second inequality holds since

$$\begin{aligned} \frac{\binom{n}{k}}{\binom{n-1}{l-1} - \binom{n-2}{l-2}} &= \frac{\binom{n}{k}}{\binom{n-1}{l-2}} = \frac{n}{n-k} \frac{\binom{n-1}{k}}{\binom{n-1}{l-2}} \leq 2 \frac{(l-2)(l-3) \dots (k+1)}{(n-k-1)(n-k-2) \dots (n-l+2)} \\ &\leq 2 \left(\frac{l-2}{n-k-1} \right)^{l-k-2} \leq 2 \left(\frac{l-2}{l+\eta l+c-1} \right)^{c-3} \leq \frac{1}{C}, \end{aligned}$$

provided c is sufficiently large depending on η and C . Hence, we may assume that

$$\binom{n-2}{l-2} \leq |\mathcal{A}| \leq \binom{n-1}{l-1}.$$

Therefore, since \mathcal{A} is lexicographically ordered, we have $\mathcal{A} \supset \{S \in [n]^{(l)} : 1, 2 \in S\}$. Hence, $B \cap \{1, 2\} \neq \emptyset$ for all $B \in \mathcal{B}$. (If there exists $B \in \mathcal{B}$ with $B \cap \{1, 2\} = \emptyset$, then since $n \geq k+l$, there exists $A \in [n]^{(l)}$ with $A \cap B = \emptyset$ and $1, 2 \in A$, but the latter implies $A \in \mathcal{A}$, a contradiction.) Therefore, since \mathcal{B} is lexicographically ordered, we have $\mathcal{B} \supset \mathcal{F}_1^{(k)}$.

Observe that

$$\mathcal{A}_{\{1,2\}}^{\{1\}} \subseteq ([n] \setminus [2])^{(l-1)}, \quad \mathcal{B}_{\{1,2\}}^{\{2\}} \subseteq ([n] \setminus [2])^{(k-1)}$$

are cross-intersecting, and trivially $|\mathcal{A}_{\{1,2\}}^{\{1\}}| \leq \binom{n-2}{l-1}$. Hence, by the induction hypothesis (which we may apply since $(n-2) \geq (1+\eta)(l-1) + (k-1) + c$ and $l-1 \geq k-1 + c-1$, choosing $d = n-2$), we have

$$|\mathcal{A}_{\{1,2\}}^{\{1\}}| + C |\mathcal{B}_{\{1,2\}}^{\{2\}}| \leq \binom{n-2}{l-1},$$

and therefore,

$$\begin{aligned} |\mathcal{A}| + C |\mathcal{B}| &= \binom{n-2}{l-2} + |\mathcal{A}_{\{1,2\}}^{\{1\}}| + C \binom{n-1}{k-1} + C |\mathcal{B}_{\{1,2\}}^{\{2\}}| \\ &\leq \binom{n-2}{l-2} + \binom{n-2}{l-1} + C \binom{n-1}{k-1} \\ &= \binom{n-1}{l-1} + C \binom{n-1}{k-1} \\ &\leq \binom{n}{l} - \binom{n-d}{l} + C \binom{n-d}{k-d}, \end{aligned}$$

using (32) for the last inequality. This completes the proof. \square

We now give a corollary of Lemma 4.7, with a choice of parameters which will be useful later.

Corollary 4.8. *For any $\eta > 0$ and any $t \in \mathbb{N}$, there exists $c = c(\eta, t) \in \mathbb{N}$ such that the following holds. Let $k, n, d \in \mathbb{N}$ with $n > (2 + \eta)k + c$ and $d \geq c$. Suppose that*

$$\mathcal{C} \subset ([n] \setminus [t])^{(k-t)}, \quad \mathcal{D} \subset ([n] \setminus [t])^{(k+1-t)}$$

are cross-intersecting. Suppose also that

$$\binom{n-t}{k-t} - \binom{n-t-c}{k-t} \leq |\mathcal{C}| \leq \binom{n-t}{k-t} - \binom{n-t-d}{k-t}.$$

Then

$$|\mathcal{C}| + (2^t - 1) |\mathcal{D}| \leq \binom{n-t}{k-t} - \binom{n-t-d}{k-t} + (2^t - 1) \binom{n-t-d}{k-t-d+1}.$$

Proof. By Lemma 4.6, we may assume that \mathcal{C} and \mathcal{D} are lexicographically ordered, so that in particular $\mathcal{C} \supset \text{OR}_{\{t+1, \dots, t+c\}} \cap ([n] \setminus [t])^{(k-t)}$. Therefore,

$$|\mathcal{C}| = \binom{n-t}{k-t} - \binom{n-t-c}{k-t} + |\mathcal{C}_{\{t+1, \dots, t+c\}}^\emptyset|.$$

Additionally (using $n > 2k$) we have that all the sets in \mathcal{D} contain $\{t+1, \dots, t+c\}$. Now note that

$$\mathcal{C}_{\{t+1, \dots, t+c\}}^\emptyset \subset ([n] \setminus [t+c])^{(k-t)}, \quad \mathcal{D}_{\{t+1, \dots, t+c\}}^{\{t+1, \dots, t+c\}} \subset ([n] \setminus [t+c])^{(k+1-t-c)}$$

are cross-intersecting. Using Lemma 4.7, with $n' = n - t - c$, $k' = k - t - c + 1$, $l' = k - t$ and $d' = d - c$, $C' = 2^t - 1$, where $c := c_0(\eta, C') = c_0(\eta, 2^t - 1)$, we obtain

$$\begin{aligned} |\mathcal{C}| + (2^t - 1) |\mathcal{D}| &= \binom{n-t}{k-t} - \binom{n-t-c}{k-t} + |\mathcal{C}_{\{t+1, \dots, t+c\}}^\emptyset| + (2^t - 1) |\mathcal{D}_{\{t+1, \dots, t+c\}}^{\{t+1, \dots, t+c\}}| \\ &\leq \binom{n-t}{k-t} - \binom{n-t-c}{k-t} + \\ &\quad + \binom{n-t-c}{k-t} - \binom{n-t-d}{k-t} + (2^t - 1) \binom{n-t-d}{k-t-d+1} \\ &= \binom{n-t}{k-t} - \binom{n-t-d}{k-t} + (2^t - 1) \binom{n-t-d}{k-t-d+1}. \end{aligned}$$

□

Now we are ready to prove our bootstrapping lemma.

Proposition 4.9. *For any $\eta > 0$ and any $t \in \mathbb{N}$, there exists $c = c_1(\eta, t) \in \mathbb{N}$ such that the following holds. Let $k, n, d \in \mathbb{N}$ with $n > (2 + \eta)k + c$ and $d \geq c$, let $\mathcal{A} \subset [n]^{(k)}$ be a t -intersecting family, and let $B \subset [n]^{(t)}$. Suppose that*

$$\binom{n-t}{k-t} - \binom{n-t-c}{k-t} \leq |\mathcal{A} \cap \mathcal{S}_B| \leq \binom{n-t}{k-t} - \binom{n-t-d}{k-t}.$$

Then

$$|\mathcal{A}| \leq \binom{n-t}{k-t} - \binom{n-t-d}{k-t} + (2^t - 1) \binom{n-t-d}{k-t-d+1}.$$

Proof. Denote $\mathcal{C} := \mathcal{A}_B^B$. Let $C_0 \subsetneq B$ be such that $|\partial^{t-|C_0|-1}(\mathcal{A}_B^{C_0})|$ is maximal amongst all $C \subsetneq B$. Using Theorem 2.10 (which can be applied, since for any $C \subsetneq B$, \mathcal{A}_B^C is $(t-|C|)$ -intersecting), we have

$$|\mathcal{A}| = |\mathcal{C}| + \sum_{C \subsetneq B} |\mathcal{A}_B^C| \leq |\mathcal{C}| + \sum_{C \subsetneq B} |\partial^{t-|C|-1}(\mathcal{A}_B^C)| \leq |\mathcal{C}| + (2^t - 1) |\partial^{t-|C_0|-1}(\mathcal{A}_B^{C_0})|. \quad (33)$$

As \mathcal{C} and $\mathcal{D} = \partial^{t-|C_0|-1}(\mathcal{A}_B^{C_0})$ are cross-intersecting, the assertion follows from (33) by applying Corollary 4.8 to \mathcal{C} and \mathcal{D} . \square

4.1.3 Proof of Theorem 1.5

Theorem 1.5 follows easily from Propositions 4.4 and 4.9.

Proof of Theorem 1.5. Let n, k, t and η be as in the statement of the theorem. Let $\delta_0 = \delta_0(\eta, t) > 0$ to be chosen later. By the equality case of Theorem 1.3, we may assume throughout that $n \geq n_0(\eta, t)$ for any $n_0(\eta, t) \in \mathbb{N}$, by choosing δ_0 to be sufficiently small.

Let $\mathcal{A} \subseteq [n]^{(k)}$ be a t -intersecting family with

$$|\mathcal{A}| > \max \left\{ (1 - \delta_0) \binom{n-t}{k-t}, \binom{n-t}{k-t} - \binom{n-t-d}{k-t} + (2^t - 1) \binom{n-t-d}{k-t-d+1} \right\}.$$

Choose $\epsilon = \epsilon(\eta, t) > 0$ such that

$$\binom{n-t}{k-t} (1 - 2\epsilon) \geq \binom{n-t}{k-t} - \binom{n-t-c}{k-t},$$

where $c = c_1(\eta, t)$ is as given by Proposition 4.9. Let $\delta = \delta(\epsilon, \eta, t) > 0$ be as given by Proposition 4.4. By reducing δ if necessary, we may assume that $\delta \leq \epsilon$. Provided $\delta_0 \leq \delta$, by Proposition 4.4 there exists $B \in [n]^{(t)}$ such that

$$|\mathcal{A} \setminus \mathcal{S}_B| \leq \epsilon \binom{n-t}{k-t},$$

and therefore

$$|\mathcal{A} \cap \mathcal{S}_B| \geq (1 - \delta - \epsilon) \binom{n-t}{k-t} \geq (1 - 2\epsilon) \binom{n-t}{k-t} \geq \binom{n-t}{k-t} - \binom{n-t-c}{k-t}.$$

Suppose for a contradiction that

$$|\mathcal{A} \cap \mathcal{S}_B| \leq \binom{n-t}{k-t} - \binom{n-t-d}{k-t}.$$

It follows that $d \geq c$. Provided $n \geq n_0(\eta, t)$ for some sufficiently large $n_0(\eta, t) \in \mathbb{N}$, our assumption that $k \leq (1/(t+1) - \eta)n$ implies that $n > (2 + \eta)k + c$, and so by Proposition 4.9, we have

$$|\mathcal{A}| \leq \binom{n-t}{k-t} - \binom{n-t-d}{k-t} + (2^t - 1) \binom{n-t-d}{k-t-d+1},$$

a contradiction. Hence,

$$|\mathcal{A} \cap \mathcal{S}_B| > \binom{n-t}{k-t} - \binom{n-t-d}{k-t}.$$

Therefore, by Lemma 4.3, we have

$$|\mathcal{A} \setminus \mathcal{S}_B| \leq (2^t - 1) \binom{n-t-d}{k-t-d+1},$$

as required. \square

4.2 A k -uniform stability result for triangle-intersecting families of graphs

To obtain a k -uniform analogue of Corollary 3.18, we just need the following analogue of Proposition 4.4.

Lemma 4.10. *For any $\eta, \epsilon > 0$, there exist $\delta = \delta(\epsilon, \eta) > 0$ and $n_0 = n_0(\epsilon, \eta) \in \mathbb{N}$ such that the following holds. Let $k, n \in \mathbb{N}$ with $n \geq n_0$ and $k \leq (\frac{1}{2} - \eta)\binom{n}{2}$. Let $\mathcal{A} \subset ([n]^{(2)})^{(k)}$ be a triangle-intersecting family of k -edge graphs with vertex-set $[n]$, such that*

$$|\mathcal{A}| \geq (1 - \delta) \binom{\binom{n}{2} - 3}{k - 3}.$$

Then there exists a triangle T such that

$$|\mathcal{A} \setminus \mathcal{S}_T| \leq \epsilon \binom{\binom{n}{2} - 3}{k - 3}.$$

Proof. The proof is almost exactly the same as the $t = 3$ case of the proof of Proposition 4.4 (applied with $\binom{n}{2}$ in place of n), except that we choose $p_0 = 1/2$ instead of $p_0 = 1/(t+1) = 1/4$, and apply Corollary 3.18 instead of Theorem 3.17; the details are omitted. \square

Applying Lemma 4.10, and the $t = 3$ case of Proposition 4.9 (with $\binom{n}{2}$ in place of n), we get the following.

Theorem 4.11. *For any $\eta > 0$, there exist $\delta_0 = \delta_0(\eta) > 0$ and $n_0 = n_0(\eta) \in \mathbb{N}$ such that the following holds. Let $k, n \in \mathbb{N}$ with $n \geq n_0$ and $k \leq (\frac{1}{2} - \eta)\binom{n}{2}$, and let $d \in \mathbb{N}$. Let $\mathcal{A} \subset ([n]^{(2)})^{(k)}$ be a triangle-intersecting family of k -edge graphs with vertex-set $[n]$, such that*

$$|\mathcal{A}| > \max \left\{ \binom{\binom{n}{2} - 3}{k - 3} (1 - \delta_0), \binom{\binom{n}{2} - 3}{k - 3} - \binom{\binom{n}{2} - d - 3}{k - 3} + 7 \binom{\binom{n}{2} - d - 3}{k - d - 2} \right\}.$$

Then there exists a triangle T such that

$$|\mathcal{A} \setminus \mathcal{S}_T| \leq 7 \binom{\binom{n}{2} - d - 3}{k - d - 2}.$$

Note that this result is stronger than the stability theorem for triangle-intersecting families of k -edge graphs presented in [18].

4.3 A k -uniform stability result for the Erdős matching conjecture

In this subsection we prove the following stability result for the Erdős matching conjecture; this can be seen as a stability version of Frankl's theorem (Theorem 3.21).

Theorem 4.12. *For any $s \in \mathbb{N}$, $\eta > 0$ and $\epsilon > 0$, there exists $\delta = \delta(\epsilon, s, \eta) > 0$ such that the following holds. Let $n, k \in \mathbb{N}$ with $k \leq (\frac{1}{2s+1} - \eta)n$. Suppose $\mathcal{A} \subset [n]^{(k)}$ with $m(\mathcal{A}) \leq s$ and*

$$|\mathcal{A}| \geq \binom{n}{k} - \binom{n-s}{k} - \delta \binom{n-s}{k-1}.$$

Then there exists $B \subset [n]^{(s)}$ such that

$$|\mathcal{A} \setminus \text{OR}_B| \leq \epsilon \binom{n-s}{k}.$$

Proof. By the equality case of Theorem 3.21, we have $|\mathcal{A}| = \binom{n}{k} - \binom{n-s}{k}$ only if $\mathcal{A} = \text{OR}_B \cap [n]^{(k)}$ for some $B \in [n]^{(s)}$, and thus, by making δ smaller if necessary, we may assume throughout that $n \geq n_0(\epsilon, s, \eta)$ for any $n_0(\epsilon, s, \eta) \in \mathbb{N}$. Choose $\delta = \delta(\epsilon, s, \eta) > 0$ such that

$$\delta \binom{n-s}{k-1} \leq \binom{n-s-c}{k-1}$$

for all $n \geq n_0$ and all $k \leq n/(2s+1)$, where $c = c(\epsilon, s, \eta) \in \mathbb{N}$ is to be chosen later.

Suppose $\mathcal{A} \subset [n]^{(k)}$ with $m(\mathcal{A}) \leq s$ and

$$|\mathcal{A}| \geq \binom{n}{k} - \binom{n-s}{k} - \delta \binom{n-s}{k-1}.$$

Then

$$|\mathcal{A}| \geq \binom{n}{k} - \binom{n-s}{k} - \binom{n-s-c}{k-1}.$$

Let $\mathcal{F} = \mathcal{A}^\uparrow$, and define

$$\mathcal{C} := \{C \subset [n] : C \cap [s-1] \neq \emptyset\} \cup \{C \subset [n] : C \cap [s] = \{s\}, C \cap \{s+1, \dots, s+c\} \neq \emptyset\}.$$

We have

$$|\mathcal{F}^{(k)}| = |\mathcal{A}| \geq \binom{n}{k} - \binom{n-s}{k} - \binom{n-s-c}{k-1} = |\mathcal{C}^{(k)}|.$$

Since $\mathcal{C}^{(k)}$ is an initial segment of the lexicographic ordering on $[n]^{(k)}$, by the Kruskal-Katona theorem

$$|\mathcal{F}^{(l)}| \geq |\mathcal{C}^{(l)}| \quad \forall l \geq k.$$

Define $p_1 := k/n$, $p_0 := 1/(2s+1)$, and $p := \frac{1}{2}(p_0 + p_1)$. By the Chernoff bound in Proposition 4.1, we have

$$\mu_p(\{F \subset [n] : |F| < k\}) = \Pr\{\text{Bin}(n, p) < k\} = o_{\eta, s}(1),$$

and thus,

$$\begin{aligned} \mu_p(\mathcal{F}) &\geq \mu_p(\{F \subset [n] : |F| \geq k\}) \mu_p(\{F \in \mathcal{F} : |\{F\}| \geq k\}) \geq (1 - o_{\eta, s}(1)) \mu_p(\mathcal{C}) \\ &= (1 - o_{\eta, s}(1))(1 - (1-p)^s - p(1-p)^{s+c-1}) = 1 - (1-p)^s - p(1-p)^{s+c-1} - o_{\eta, s}(1). \end{aligned}$$

Since $m(\mathcal{F}) = m(\mathcal{A}) \leq s$, we have $\mu_{p_0}(\mathcal{F}) \leq 1 - (1-p_0)^s$. Hence, by Corollary 3.23, if

$$(1-p)^c + o_{\eta, s}(1) \leq \frac{1-p}{p} \left(\frac{1}{2s+1} - p \right) (1 - (1-p)^s), \quad (34)$$

then there exists $B \in [n]^{(s)}$ such that

$$\mu_p(\mathcal{F} \setminus \text{OR}_B) \leq (1-p)^s \epsilon_1,$$

where $\epsilon_1 > 0$ is the minimal positive solution to

$$(1-p)^c + o_{\eta, s}(1) = \tilde{c} \epsilon_1^{\log_p(1/(2s+1)) \log_{2s/(2s+1)}(1-p)} - \frac{1-p}{p} \epsilon_1,$$

and $\tilde{c} := (2s)^{\log_{2s/(2s+1)}(1-p)}$. Provided c and n are sufficiently large depending on η and s , (34) does indeed hold, and crudely, we have

$$\epsilon_1 = O_{s,\eta}(1)(1-p)^c + o_{\eta,s}(1),$$

and therefore

$$\mu_p(\mathcal{F} \setminus \text{OR}_B) \leq (1-p)^s(O_{s,\eta}(1)(1-p)^c + o_{\eta,s}(1)). \quad (35)$$

Without loss of generality, we may assume that $B = [s]$. Choose $d \in \mathbb{N}$ minimal such that

$$\binom{n-s-d}{k-d} \leq \epsilon \binom{n-s}{k};$$

then $d < d_0(s, \epsilon)$ for all $k \leq n/(2s+1)$. Suppose for a contradiction that

$$|\mathcal{A} \setminus \text{OR}_B| > \epsilon \binom{n-s}{k}.$$

Then

$$|(\mathcal{F}_{[s]}^\emptyset)^{(k)}| = |\mathcal{A}_{[s]}^\emptyset| = |\mathcal{A} \setminus \text{OR}_B| > \epsilon \binom{n-s-d}{k-d}.$$

Let $\mathcal{D} := \{D \subset [n] \setminus [s] : \{s+1, \dots, s+d\} \subset D\}$. By the Kruskal-Katona theorem, we have

$$|(\mathcal{F}_{[s]}^\emptyset)^{(l)}| \geq \binom{n-s-d}{l-d} = |\mathcal{D}^{(l)}| \quad \forall l \geq k.$$

By the Chernoff bound in Proposition 4.1, we have

$$\mu_p(\{F \subset [n] \setminus [s] : |F| < k\}) = \Pr\{\text{Bin}(n-s, p) < k\} = o_{\eta,s}(1),$$

whence

$$\begin{aligned} \mu_p(\mathcal{F}_{[s]}^\emptyset) &\geq \mu_p(\{F \subset [n] \setminus [s] : |F| \geq k\}) \mu_p(\{F \in \mathcal{F}_{[s]}^\emptyset : |F| \geq k\}) \geq (1 - o_{\eta,s}(1)) \mu_p(\mathcal{D}) \\ &= (1 - o_{\eta,s}(1)) p^d = p^d - o_{\eta,s}(1). \end{aligned}$$

Therefore,

$$\mu_p(\mathcal{F} \setminus \text{OR}_{[s]}) \geq (1-p)^s(p^d - o_{\eta,s}(1)).$$

This contradicts (35), provided c is sufficiently large depending on s, η and $d_0(s, \epsilon)$, and n is sufficiently large depending on s and η . This completes the proof. \square

Remark 4.13. In Theorem 4.12, the relation between δ and ϵ is not specified. Currently, we are able to prove (by a more complex argument) that the theorem holds with $\epsilon = (c\delta)^{\log_{1-sp_1} p_1}$, where $c = c(s, \eta) > 0$, and $p_1 := k/n$. However, we believe that the right dependence is $\epsilon = (c\delta)^{\log_{1-p_1} p_1}$; this would follow from Conjecture 6.3.

5 A comparison with some prior results

A central feature of several of our results is replacement of the (t) -intersection assumption of EKR-type theorems with a weaker assumption (specifically, an upper bound on $\mu_{p_0}(\mathcal{F})$ for some p_0). In this section, we compare our results with some prior results on stability for intersection problems.

5.1 Intersecting families

One of the strongest known stability results for the EKR theorem is Frankl's theorem from 1987 [23], briefly mentioned in the introduction. To state it in full, we need some more definitions.

If $\mathcal{F} \subset \mathcal{P}([n])$, we define $\deg(\mathcal{F}) := \max_{j \in [n]} |\{F \in \mathcal{F} : j \in F\}|$. For $2 \leq k \leq n-1$ and $3 \leq i \leq k+1$, we define

$$\mathcal{G}_i := \{A \in [n]^{(k)} : (1 \in A) \wedge (A \cap \{2, 3, \dots, i\} \neq \emptyset)\} \cup \{A \in [n]^{(k)} : (1 \notin A) \wedge (\{2, 3, \dots, i\} \subset A)\}.$$

Clearly, each \mathcal{G}_i is an intersecting family.

Theorem 5.1 (Frankl, 1987). *Let $n, k, i \in \mathbb{N}$ with $n > 2k$ and $3 \leq i \leq k+1$. Let $\mathcal{F} \subset [n]^{(k)}$ be an intersecting family with $\deg(\mathcal{F}) \leq \deg(\mathcal{G}_i)$. Then $|\mathcal{F}| \leq |\mathcal{G}_i|$.*

Theorem 5.1 implies the following biased-measure version, via the method of ‘going to infinity and back’. For $3 \leq i \leq n$, we define

$$\tilde{\mathcal{G}}_i := \{A \subset \mathcal{P}([n]) : (1 \in A) \wedge (A \cap \{2, 3, \dots, i\} \neq \emptyset)\} \cup \{A \subset \{0, 1\}^n : (1 \notin A) \wedge (\{2, 3, \dots, i\} \subset A)\}.$$

Clearly, we have

$$\mu_p(\tilde{\mathcal{G}}_i) = p(1 - (1-p)^{i-1}) + (1-p)p^{i-1}, \quad \mu_p(\tilde{\mathcal{G}}_i \setminus \mathcal{F}_1) = (1-p)p^{i-1}.$$

Corollary 5.2. *Suppose that $0 < p < 1/2$ and $3 \leq i \leq n$. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an intersecting family with $\mu_p(\mathcal{F}) > \mu_p(\tilde{\mathcal{G}}_i) = p(1 - (1-p)^{i-1}) + (1-p)p^{i-1}$. Then there exists a dictatorship \mathcal{F}_j such that $\mu_p(\mathcal{F} \cap \mathcal{F}_j) > \mu_p(\tilde{\mathcal{G}}_i \cap \mathcal{F}_1) = p(1 - (1-p)^{i-1})$.*

An application of Lemma 2.9, together with the observation that $\mathcal{F}_{\{j\}}^\emptyset$ and $\mathcal{F}_{\{j\}}^{\{j\}}$ are cross-intersecting for all $j \in [n]$, yields the following.

Corollary 5.3. *Suppose that $0 < p < 1/2$ and $3 \leq i \leq n$. Let $\mathcal{F} \subset \mathcal{P}([n])$ be an intersecting family with $\mu_p(\mathcal{F}) > \mu_p(\tilde{\mathcal{G}}_i) = p(1 - (1-p)^{i-1}) + (1-p)p^{i-1}$. Then there exists a dictatorship \mathcal{F}_j such that $\mu_p(\mathcal{F} \setminus \mathcal{F}_j) < \mu_p(\tilde{\mathcal{G}}_i \setminus \mathcal{F}_1) = (1-p)p^{i-1}$.*

Comparison of our Theorem 1.8 with the corollary of Frankl's result (i.e., Corollary 5.3), shows that the case $\epsilon = p^{i-1}$ of the Frankl corollary implies the same case of our corollary. On the one hand, the Frankl corollary has an important advantage over ours: it applies whenever $\mu_p(\mathcal{F}) > \mu_p(\tilde{\mathcal{G}}_3) = 3p^2 - 2p^3$. Theorem 1.8 applies only under the condition (1), i.e. for $\mu_p(\mathcal{F}) \geq Cp^2$ (for a sufficiently large absolute constant C) when p is small, and for $\mu_p(\mathcal{F}) \geq p(1 - c(1/2 - p))$ (for a sufficiently small absolute constant c) when p is large. On the other hand, Theorem 1.8 has two advantages over Frankl's: firstly, we only assume that \mathcal{F} is increasing and that $\mu_{1/2}(\mathcal{F}) \leq 1/2$, which is weaker than the intersection assumption of Frankl. Secondly, for any ϵ which is not of the form p^{i-1} , our result is stronger than Frankl's, provided the condition (1) holds. In [17] we present a combinatorial argument showing that Theorem 1.8 holds when the condition (1) is replaced by the ‘Frankl’ condition $\mu_p(\mathcal{F}) > 3p^2 - 2p^3$, providing a ‘strict’ improvement of Corollary 5.3, which applies to any increasing $\mathcal{F} \subset \mathcal{P}([n])$ with $\mu_{1/2}(\mathcal{F}) \leq 1/2$.

5.2 t -intersecting families, for $t > 1$

Ahlswede and Khachatrian obtained in [2] a stability result for the AK theorem which applies to families of size *very* close to the maximum. However, the only previously known stability result for Wilson's theorem which applies for families of size within a constant fraction of the maximum, is Friedgut's Theorem 1.4. Our Theorem 1.5 implies a strengthening of Theorem 1.4, with $\epsilon^{\log_{1-k/n}(k/n)}$ replacing ϵ in the conclusion of Theorem 1.4, i.e. sharp ϵ -dependence. It is interesting to note that unlike the proof of Friedgut's theorem, our proof of Theorem 1.5 does not rely upon neither Fourier analysis nor spectral techniques.

5.3 The Erdős matching conjecture

Frankl (unpublished; see [24]) has proved the following Hilton-Milner type result for the Erdős matching conjecture. If $x_0, x_1, \dots, x_{s-1} \in [n]$ are distinct, and $T_1, \dots, T_s \in [n]^{(k)}$ are pairwise disjoint with $x_i \in T_i$ for all $i \in [s-1]$ and $x_0 \notin T_1 \cup T_2 \cup \dots \cup T_s$, we define the family

$$\mathcal{E}(n, k, s) = \{S \in [n]^{(k)} : \exists i \in \{0\} \cup [s-1] \text{ such that } x_i \in S, S \cap (T_{i+1} \cup \dots \cup T_s) \neq \emptyset\} \\ \cup \{T_1, \dots, T_s\}.$$

Theorem 5.4 (Frankl). *For any $k, s \in \mathbb{N}$ with $k \geq 4$, there exists $n_1 = n_1(k, s) \in \mathbb{N}$ such that the following holds. Let $n \in \mathbb{N}$ with $n \geq n_1$, and let $\mathcal{F} \subset [n]^{(k)}$ with $m(\mathcal{F}) = s$ and $m(\mathcal{F}_{\{i\}}^\emptyset) = s$ for all $i \in [n]$. Then $|\mathcal{F}| \leq |\mathcal{E}(n, k, s)|$, with equality if and only if \mathcal{F} is isomorphic to $\mathcal{E}(n, k, s)$.*

He has also proved a similar result for $k = 3$ (unpublished; see [24]).

Kostochka and Mubayi [43] have recently proved another stability result for the Erdős matching conjecture (Theorem 10 in [43]), together with several more stability results for Erdős-Ko-Rado type problems. Their proofs rely on Frankl's delta-system method, and their results therefore only apply for $n \geq n_1(k, s)$, where $n_1(k, s)$ is at least exponential in k (for each $s \in \mathbb{N}$). In this range, their results do not imply ours and are not implied by ours.

To the best of our knowledge, Theorem 4.12 is the first stability result for the Erdős matching conjecture which applies when $k = \Theta(n)$ (indeed, whenever k/n is bounded away from $\frac{1}{2s+1}$).

6 Problems for further research

Tighter stability for EKR-type problems. In Theorem 3.1, the relation we obtain between $\mu_p(\mathcal{F})$ and $\mu_p(\mathcal{F} \setminus \mathcal{S}_B)$ is tight. However, in all applications where the assumptions involve intersection properties (e.g., Theorems 3.17, 4.12 and Corollaries 3.18, 3.23) we believe that our results are not tight in some of the parameters. We conjecture the following strengthening of Theorem 3.17.

Conjecture 6.1. *Let $t \in \mathbb{N}$, let $0 < p < 1/(t+1)$, and let $\mathcal{F} \subset \mathcal{P}([n])$ be a t -intersecting family such that*

$$\mu_p(\mathcal{F}) \geq (t+2)p^{t+1} - (t+1)p^{t+2}.$$

Let $\epsilon > 0$. If

$$\mu_p(\mathcal{F}) \geq p^t \left(1 - \left(\frac{\epsilon}{t} \right)^{\log_p(1-p)} \right) + (1-p)p^{t-1}\epsilon,$$

then there exists a t -umvirate \mathcal{S}_B such that $\mu_p(\mathcal{F} \setminus \mathcal{S}_B) \leq (1-p)p^{t-1}\epsilon$.

This would be sharp, as evidenced by the families $\{\tilde{\mathcal{F}}_{t,s}\}_{t,s \in \mathbb{N}}$ defined by

$$\begin{aligned} \tilde{\mathcal{F}}_{t,s} := & \{A \subset \mathcal{P}([n]) : [t] \subset A, \{t+1, \dots, t+s\} \cap A \neq \emptyset\} \\ & \cup \{A \subset \mathcal{P}([n]) : |[t] \cap A| = t-1, \{t+1, \dots, t+s\} \subset A\}. \end{aligned}$$

Likewise, we conjecture the following strengthening of Theorem 1.5.

Conjecture 6.2. *Let $t, k, n \in \mathbb{N}$ such that $n \geq (t+1)(k-t+1)$, and let $d \in \mathbb{N}$. If $\mathcal{A} \subset [n]^{(k)}$ is a t -intersecting family with*

$$\begin{aligned} |\mathcal{A}| \geq & \max\left\{(t+2)\binom{n-t-2}{k-t-1} - (t+1)\binom{n-t-2}{k-t-2}, \right. \\ & \left. \binom{n-t}{k-t} - \binom{n-t-d}{k-t} + t\binom{n-t-d}{k-t-d+1}\right\}, \end{aligned}$$

there exists a t -umvirate \mathcal{S}_B such that

$$|\mathcal{A} \setminus \mathcal{S}_B| \leq t\binom{n-t-d}{k-t-d+1}.$$

This would be sharp, as evidenced by the families $\{\tilde{\mathcal{F}}_{t,s} \cap [n]^{(k)}\}_{t,s \in \mathbb{N}}$ (which are the families $\mathcal{F}_{t,s}$ defined in the introduction).

It would also be of interest to prove tight versions of Corollary 3.18 and of Theorem 4.11, regarding triangle-intersecting families of graphs.

It seems that additional tools which exploit the intersection properties more fully, would be needed to prove the above.

Stability for families with measure not-so-close to the maximum. In the case $p_0 = 1/2$, we conjecture that the condition (7) in Theorem 3.1 could be replaced by the condition $\mu_p(\mathcal{F}) \geq (t+2)p^{t+1} - (t+1)p^{t+2} = \mu_p(\{S \subset [n] : |S \cap [t+2]| \geq t+1\})$. In [17] we prove this for $t = 1$, by adapting the techniques of Frankl in [23], but these techniques do not extend to the case of $t > 1$. It would also be of interest to determine, for each $p_0 \in (0, 1)$, the sharp analogue of the condition (7) in Theorem 3.1.

Stability in cases where the extremal example is not a t -umvirate or its dual. It seems that the techniques used in this paper are applicable only in cases where the extremal family (corresponding to $\epsilon = 0$), or its dual, is a family for which equality holds in the biased edge-isoperimetric inequality on the hypercube (Theorem 2.2), i.e., the extremal family must be a t -umvirate or its dual. It would be interesting to see whether these techniques can be adapted to cases where the extremal family is more complex, e.g., the full Ahlswede-Khachatrian theorem, where the extremal families are isomorphic to $\mathcal{F}_{n,k,t,r} = \{S : |S \cap [t+2r]| \geq t+r\}$. We note that in a recent work [41], the two last named authors established a stability version of the full AK theorem. However, the techniques used in [41] are rather different from those we use here, and the results obtained there do not imply the results presented here.

Sharp stability for the Erdős matching conjecture. The most obvious open question in this area is to resolve the Erdős matching conjecture. We also conjecture the following strengthening of Theorem 4.12.

Conjecture 6.3. Let $s, k, n \in \mathbb{N}$ with $n \geq (s+1)k$. Let $d \in \mathbb{N}$, and suppose $\mathcal{F} \subset [n]^{(k)}$ is such that $m(\mathcal{F}) = s$ and

$$|\mathcal{F}| \geq \max \left\{ \binom{k(s+1)-1}{k} + 1, \binom{n}{k} - \binom{n-s}{k} - \binom{n-s-d}{k-1} + \binom{n-s-d}{k-d} \right\}.$$

Then there exists $B \in [n]^{(s)}$ such that

$$|\mathcal{A} \setminus \text{OR}_B| \leq \binom{n-s-d}{k-d}.$$

This would be sharp, as evidenced by the family

$$\begin{aligned} \mathcal{H} = & \{A \in [n]^{(k)} : A \cap [s-1] \neq \emptyset\} \\ & \cup \{A \in [n]^{(k)} : A \cap [s] = \{s\}, A \cap \{s+1, s+2, \dots, s+d\} \neq \emptyset\} \\ & \cup \{A \in [n]^{(k)} : A \cap [s] = \emptyset, \{s+1, \dots, s+d\} \subset A\}. \end{aligned}$$

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